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LAGRANGIAN PRODUCT TORI IN TAME SYMPLECTIC MANIFOLDS

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Abstract. In [3], product Lagrangian tori in standard symplectic space $\mathbb{R}^{2n}$ were classified up to symplectomorphism. We extend this classification to tame symplectically aspherical symplectic manifolds. We show by examples that the asphericity assumption cannot be omitted.

1. Introduction and main results

Let $T(a)$ denote the boundary of the disc of area $a > 0$ in $\mathbb{R}^2$ centred at the origin. Let $\boldsymbol{a} = (a_1, \dots, a_n)$ be a vector with positive components. We call the n-torus

$$T(\boldsymbol{a}) = T(a_1) \times \cdots \times T(a_n) \subset \mathbb{R}^{2n}$$

a *product torus*. Product tori are Lagrangian with respect to the standard symplectic form $\omega_n = \sum_{j=1}^n dx_j \wedge dy_j$, that is, the restriction of ω_n to each product torus vanishes.

Let (M, ω) be a symplectic manifold. We assume throughout the paper that M is connected. Denote by $B^{2n}(b)$ the closed ball of radius $\sqrt{b/\pi}$ in $\mathbb{R}^{2n}$ centred at the origin. The torus $T(\boldsymbol{a})$ lies on the boundary of the ball $B^{2n}(|\boldsymbol{a}|)$. By a *symplectic chart* we understand a symplectic embedding $\varphi\colon B^{2n}(b) \to (M, \omega)$. Given a symplectic chart $\varphi\colon B^{2n}(b) \to (M, \omega)$ and a torus $T(\boldsymbol{a}) \subset B^{2n}(b)$, we write $T_\varphi(\boldsymbol{a}) = \varphi(T(\boldsymbol{a}))$. A Lagrangian torus in (M, ω) is called a *product torus* if it is of the form $T_\varphi(\boldsymbol{a})$ for some symplectic chart φ.

We study the classification problem for product Lagrangian tori with respect to the action of the group $\mathrm{Symp}(M, \omega)$ of *symplectomorphisms* of M (diffeomorphisms preserving the symplectic form ω) as well as the group $\mathrm{Ham}(M, \omega)$ of *Hamiltonian symplectomorphisms*. Hamiltonian symplectomorphisms are defined as follows. Let $\{H_t\}$ be a family of smooth functions on M smoothly depending on the parameter $t \in [0, 1]$. This family defines a family of Hamiltonian vector fields $\{X_t\}$ by $\omega(X_t, \cdot) = dH_t(\cdot)$. Assume that the time t flow Ψ_t of $\{X_t\}$ is a well-defined diffeomorphism for each $t \in [0, 1]$. Then each Ψ_t is a symplectomorphism. The family $\{\Psi_t\}$ is then called a *Hamiltonian isotopy*; symplectomorphisms Ψ_t arising in this way form the subgroup $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$.

Given Lagrangian submanifolds L, L' in a symplectic manifold (M, ω), we write $L \sim L'$ (resp. $L \approx L'$) if there is a symplectomorphism (resp. a Hamiltonian symplectomorphism) of (M, ω) that maps L to L'. In the particular case where $(M, \omega) = (\mathbb{R}^{2n}, \omega_n)$, we say

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that L is Hamiltonian isotopic to L' in the ball $B^{2n}(b)$ if there is a Hamiltonian isotopy $\{\Phi_s\}$, $s \in [0,1]$, of $\mathbb{R}^{2n}$ such that $\Phi_0 = \mathtt{id}$, $\Phi_1(L) = L'$, and $\Phi_s(L) \subset B^{2n}(b)$ for all $s \in [0,1]$.

Given a vector $\boldsymbol{a} = (a_1, \ldots, a_n)$ with positive components, denote

$$\underline{\boldsymbol{a}} = \min_{1\le i\le n}(a_i), \;\; m(\boldsymbol{a}) = \#\left\{i \mid a_i = \underline{\boldsymbol{a}}\right\}, \;\; |\boldsymbol{a}| = \sum_{i=1}^{n} a_i, \;\; \|\boldsymbol{a}\| = |\boldsymbol{a}| + \underline{\boldsymbol{a}}.$$

Let $\Gamma(\boldsymbol{a})$ denote the subgroup of $\mathbb{R}$ formed by all integer combinations of the numbers $a_1 - \underline{\boldsymbol{a}}, \ldots, a_n - \underline{\boldsymbol{a}}$. We write $\boldsymbol{a} \simeq \boldsymbol{a}'$ when the following holds: $\underline{\boldsymbol{a}} = \underline{\boldsymbol{a}}'$, $m(\boldsymbol{a}) = m(\boldsymbol{a}')$, and $\Gamma(\boldsymbol{a}) = \Gamma(\boldsymbol{a}')$. It was proved in [3] that for product tori in $\mathbb{R}^{2n}$ the conditions $T(\boldsymbol{a}) \sim T(\boldsymbol{a}')$, $T(\boldsymbol{a}) \approx T(\boldsymbol{a}')$, $\boldsymbol{a} \simeq \boldsymbol{a}'$ are equivalent one to another. The following theorem gives an upper bound on the size of the support of a Hamiltonian isotopy between product tori when such an isotopy exists.

Theorem 1.1. (i) *If $\boldsymbol{a}$ and $\boldsymbol{a}'$ are related by a permutation of the components, then the tori $T(\boldsymbol{a})$ and $T(\boldsymbol{a}')$ are Hamiltonian isotopic in the ball $B^{2n}(|\boldsymbol{a}|)$.*

(ii) *If $\boldsymbol{a} \simeq \boldsymbol{a}'$, then the tori $T(\boldsymbol{a})$ and $T(\boldsymbol{a}')$ are Hamiltonian isotopic in the ball $B^{2n}\big(\max(\|\boldsymbol{a}\|, \|\boldsymbol{a}'\|)\big)$.*

Assertion (i) of the theorem is, of course, rather obvious. It seems likely that Theorem 1.1 gives a sharp bound for the ball size. However, we can only prove the sharpness under the additional assumption that $|\boldsymbol{a}| \neq |\boldsymbol{a}'|$:

Theorem 1.2. *If $b < \max(\|\boldsymbol{a}\|, \|\boldsymbol{a}'\|)$ and $|\boldsymbol{a}| \neq |\boldsymbol{a}'|$, then the tori $T(\boldsymbol{a})$ and $T(\boldsymbol{a}')$ are not Hamiltonian isotopic in the ball $B^{2n}(b)$.*

It will sometimes be necessary to assume that the geometry of the symplectic manifold (M, ω) is not too wild. Following [7, 24, 2], we say that (M, ω) is *tame* if M admits an almost complex structure J and a complete Riemannian metric g satisfying the following conditions:

(T1) J is uniformly tame, i.e., there are positive constants C_1 and C_2 such that

$$\omega\left(X, JX\right) \ge C_1 \left\|X\right\|_g^2 \quad \text{ and } \quad \left|\omega\left(X, Y\right)\right| \le C_2 \left\|X\right\|_g \left\|Y\right\|_g$$

for all tangent vectors X and Y on M.

(T2) The sectional curvature of (M, g) is bounded from above and the injectivity radius of (M, g) is bounded away from zero.

Some examples of tame symplectic manifolds are as follows: (1) closed symplectic manifolds; (2) cotangent bundles over arbitrary manifolds; (3) twisted cotangent bundles over closed manifolds; (4) symplectic manifolds such that the complement of a compact subset is symplectomorphic to the convex end of the symplectization of a closed contact manifold. The class of tame symplectic manifolds is closed under taking products and coverings.

Recall that (M, ω) is called *symplectically aspherical* if $[\omega]|_{\pi_2(M)} = 0$ and $c_1|_{\pi_2(M)} = 0$. Here, $c_1 = c_1(\omega)$ is the first Chern class of TM with respect to an (arbitrary) almost

complex structure J taming ω as in (T1), and the restriction to $\pi_2(M)$ is understood as the restriction to the image of the natural map $\pi_2(M) \to H_2(M;\mathbb{Z}) \subset H_2(M;\mathbb{R})$.

Given a symplectic chart $\varphi\colon B^{2n}(b) \to (M,\omega)$, we write $b_\varphi = b$. The following theorem shows that the invariants of product tori in $\mathbb{R}^{2n}$ extend to certain other symplectic manifolds:

Theorem 1.3. *Assume that $T_\varphi(\boldsymbol{a}) \sim T_{\varphi'}(\boldsymbol{a}')$, where $T_\varphi(\boldsymbol{a}), T_{\varphi'}(\boldsymbol{a}') \subset (M,\omega)$.*

(i) *If (M,ω) is symplectically aspherical, then $\Gamma(\boldsymbol{a}) = \Gamma(\boldsymbol{a}')$.*

(ii) *If (M,ω) is tame, $\|\boldsymbol{a}\| \le b_\varphi$, and $\|\boldsymbol{a}'\| \le b_{\varphi'}$, then $\underline{\boldsymbol{a}} = \underline{\boldsymbol{a}}'$ and $m(\boldsymbol{a}) = m(\boldsymbol{a}')$.*

A symplectic manifold (M,ω) is called a *Liouville manifold* if it admits a vector field X such that $\mathcal{L}_X\omega = \omega$ (where $\mathcal{L}_X$ is the Lie derivative with respect to X). If X can be chosen in such a way that its time t flow map is well-defined for each $t \ge 0$, we call (M,ω) *forward complete*. Examples of tame forward complete Liouville manifolds are cotangent bundles and, more generally, Stein manifolds, see [6]. Product tori in such manifolds can be completely classified:

Theorem 1.4. *Let $T_\varphi(\boldsymbol{a})$, $T_{\varphi'}(\boldsymbol{a}')$ be Lagrangian product tori in a tame forward complete Liouville manifold (M,ω). Then the conditions $\boldsymbol{a} \simeq \boldsymbol{a}'$, $T_\varphi(\boldsymbol{a}) \sim T_{\varphi'}(\boldsymbol{a}')$, $T_\varphi(\boldsymbol{a}) \approx T_{\varphi'}(\boldsymbol{a}')$ are equivalent one to another.*

The assumption $\|\boldsymbol{a}\| \le b_\varphi$, $\|\boldsymbol{a}'\| \le b_{\varphi'}$ in Theorem 1.3 (ii) cannot be omitted, as the following simple example shows. Let S^2 be the round 2-sphere, endowed with the Euclidean area form of total area 2. Let $p_N, p_S \in S^2$ be the north pole and the south pole. Choose $\varepsilon \in \,]0, \frac{1}{2}[$, and let $\varphi, \varphi'\colon B^2(2-\varepsilon) \to S^2$ be Darboux charts such that $\varphi(0) = p_N$, $\varphi'(0) = p_S$, and such that concentric circles are mapped to circles of latitude. Then $T_\varphi(\frac{1}{2}) = T_{\varphi'}(\frac{3}{2})$, but $\underline{\boldsymbol{a}} = \frac{1}{2} \neq \frac{3}{2} = \underline{\boldsymbol{a}}'$. Note that $\|\boldsymbol{a}'\| = 3 > 2 - \varepsilon = b_{\varphi'}$.

The assumption in Theorem 1.3 (i) that (M,ω) is symplectically aspherical cannot be omitted either, as the next theorem shows. Recall that the cohomology class $[\omega]$ of the symplectic form gives rise to the homomorphism $\sigma\colon \pi_2(M) \to \mathbb{R}$, and the first Chern class c_1 gives rise to the homomorphism $c_1\colon \pi_2(M) \to \mathbb{Z}$. Given $a > 0$, define the homomorphism

$$\sigma_a\colon \pi_2(M) \to \mathbb{R}, \quad S \mapsto \sigma(S) - c_1(S)a.$$

With $a > 0$ and a symplectic manifold (M,ω) we associate the group

$$G_a = G_a(M,\omega) := \sigma_a(\pi_2(M)) \subset \mathbb{R}.$$

Note that (M,ω) is symplectically aspherical if and only if G_a is trivial for all $a > 0$. We call the symplectic manifold (M,ω) *special* if the rank of the group $\sigma\big(\pi_2(M)\big) \subset \mathbb{R}$ is 1 and c_1 is not proportional to σ. We associate with each $S_0 \in \pi_2(M)$ and each $a > 0$ the subgroup $G_a(S_0) = G_a(S_0, M, \omega)$ of G_a which is the image under σ_a of the subgroup generated by S_0.

Theorem 1.5. *Let (M,ω) be a symplectic manifold; if (M,ω) is special, we also fix an element $S_0 \in \pi_2(M)$. Let $\varphi\colon B^{2n}(b) \to (M,\omega)$ be a symplectic chart. For every real number $c > 0$ there exists $A > 0$ such that for all $a \in \,]0, A]$ the following holds.*

If $d_1, \dots, d_k$ and $e_1, \dots, e_k$ for all $j \in \{1, \dots, k\}$ satisfy the conditions $d_j \geq c$, $e_j \geq c$,

$$d_j - e_j \in \begin{cases} G_a(S_0) & \text{if } (M,\omega) \text{ is special,} \\ G_a & \text{otherwise,} \end{cases}$$

and the tori $T_\varphi(a, \dots, a, a+d_1, \dots, a+d_k)$, $T_\varphi(a, \dots, a, a+e_1, \dots, a+e_k)$ are contained in $B_\varphi(b)$, then

$$T_\varphi(a, \dots, a, a+d_1, \dots, a+d_k) \approx T_\varphi(a, \dots, a, a+e_1, \dots, a+e_k).$$

Example. Given $v > 0$, we denote by $S^2(v)$ the 2-sphere of area v. There exists a symplectic embedding $B^4(b) \to S^2(v_1) \times S^2(v_2)$ whenever $b < \min(v_1, v_2)$. The homomorphism c_1 on $\pi_2(S^2(v_1) \times S^2(v_2)) = \mathbb{Z} \oplus \mathbb{Z}$ is given by $(m_1, m_2) \mapsto 2(m_1 + m_2)$. For $S_0 = (1, -1)$ we have

$$G_a\big(S_0, S^2(v_1) \times S^2(v_2)\big) = (v_1 - v_2)\,\mathbb{Z}.$$

(Note that $S^2(v_1) \times S^2(v_2)$ is special if and only if $v_1/v_2 \in \mathbb{Q}$ and $v_1 \neq v_2$.) Theorem 1.5 implies, in particular, that in $S^2(3) \times S^2(4)$ the tori $T(a, a+1)$ and $T(a, a+2)$ are Hamiltonian isotopic for all sufficiently small a, whereas $(a, a+1) \not\cong (a, a+2)$. $\diamond$

The paper is organized as follows. In Section 2, we describe the invariants that are used in the proof of Theorems 1.2 and 1.3, and derive Theorem 1.3. In Section 3 we proof a version of Theorem 1.3 for generalized Clifford tori in $\mathbb{C}\mathrm{P}^n$, and use it to prove Theorem 1.2. In Section 4 we construct Hamiltonian isotopies that provide a proof of Theorem 1.1. In Sections 5 and 6, we prove finer versions of Theorems 1.4 and 1.5, respectively. Appendix A provides a refinement of Lelong's inequality for the area of holomorphic curves passing through the centre of a ball, that we use in Section 2. Appendix B proves an algebraic result used in Section 4.

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2. Symplectic invariants

2.1. **Displacement energy and J-holomorphic discs.** The first Ekeland–Hofer capacity was a key tool used in [3] for the classification of product tori in $\mathbb{R}^{2n}$. This capacity is defined only for subsets of the standard symplectic space $\mathbb{R}^{2n}$. We shall work with the displacement energy capacity instead, which is defined for all symplectic manifolds. In the process of computing the displacement energy for Lagrangian tori, we bring J-holomorphic discs into play, and it is here that we need the assumption that (M,ω) be tame.

Consider the set $\mathcal{H}(M)$ of compactly supported smooth functions $H\colon [0,1]\times M \to \mathbb{R}$. Denote by Φ_H the time 1 map of the Hamiltonian flow generated by H. Following [9], we define a norm on $\mathcal{H}$ by

$$\|H\| = \int_0^1 \left(\max_{x\in M} H(t,x) - \min_{x\in M} H(t,x)\right) dt,$$

and the displacement energy of a compact subset $A \subset M$ by

$$e\,(A,M) = \inf_{H\in\mathcal{H}} \Big\{ \|H\| \mid \Phi_H(A)\cap A = \varnothing \Big\},$$

assuming $\inf(\varnothing)=\infty$.

Assume that (M,ω) is tame. Denote by D the closed unit disc in the complex plane $\mathbb{C}$, and by $\mathcal{J}=\mathcal{J}(M,\omega)$ the set of almost complex structures J on M for which there exists a complete Riemannian metric g such that J and g satisfy (T1) and (T2). Let L be a closed Lagrangian submanifold of (M,ω). Given $J\in\mathcal{J}$, we define $\sigma(L,M;J)$ to be the minimal symplectic area $\int_D u^*\omega$ of a non-constant J-holomorphic map $u\colon (D,\partial D)\to (M,L)$ if such maps exist, and set $\sigma(L,M;J)=\infty$ otherwise. Since (M,ω) is tame, Gromov's compactness theorem implies that the minimal area is indeed attained and thus positive [19]. Define

$$\sigma\,(L,M) = \sup_{J\in\mathcal{J}} \sigma(L,M;J),$$

allowing $\sigma(L,M)$ to be infinite as well. It was proved in [4] that

$$\sigma\,(L,M) \le e\,(L,M)\,. \tag{1}$$

Recall that $\underline{\boldsymbol{a}} = \min_{1\le i\le n}(a_i)$, $\|\boldsymbol{a}\| = \underline{\boldsymbol{a}} + \sum_{i=1}^n a_i$.

Proposition 2.1. *If (M,ω) is tame and $\|\boldsymbol{a}\|\le b_\varphi$, then $e\,(T_\varphi(\boldsymbol{a}),M)=\underline{\boldsymbol{a}}$.*

Proof. First we prove that $e\,(T_\varphi(\boldsymbol{a}),M)\le \underline{\boldsymbol{a}}$. We can assume that $a_1=\underline{\boldsymbol{a}}$. We write $D(\boldsymbol{a})$ for the polydisc $B^2(a_1)\times\cdots\times B^2(a_n)$. Let U be a neighbourhood of $B^{2n}(b)$ such that $\varphi\colon U\to M$ is well defined. Choose $\varepsilon>0$ such that $B^{2n}\big(\|\boldsymbol{a}+n\varepsilon\|\big)\subset U$. The torus $T(\boldsymbol{a})$ can be displaced from itself by the time 1 flow map of a Hamiltonian function $H\in\mathcal{H}\,(D\,(2a_1+\varepsilon, a_2+\varepsilon,\dots,a_n+\varepsilon))$ such that $\|H\|<\underline{\boldsymbol{a}}+\varepsilon$, see e.g. [10, p.171]. The polydisc $D\,(2a_1+\varepsilon,a_2+\varepsilon,\dots,a_n+\varepsilon)$ is contained in the ball $B^{2n}(\|\boldsymbol{a}\|+n\varepsilon)$ and hence in U. Transferring H to (M,ω) by means of the chart φ, we obtain a Hamiltonian function $H^\varphi\in\mathcal{H}(M)$ such that $\|H^\varphi\|<\underline{\boldsymbol{a}}+\varepsilon$ and the time 1 flow generated by H^φ disjoins the torus $T_\varphi(\boldsymbol{a})$ from itself. Since ε can be chosen arbitrarily small, it follows that $e\,(T_\varphi(\boldsymbol{a}),M)\le\underline{\boldsymbol{a}}$.

Denote by J_0 the standard complex structure on $\mathbb{C}^n$.

Lemma 2.2. *Let L be a closed Lagrangian submanifold in $B^{2n}(b_-)\subset\mathbb{C}^n$, and let φ be a symplectic chart such that $b_\varphi>b_-$. Then*

$$\sigma\big(\varphi(L),M\big) \ge \min\big(\sigma(L,\mathbb{C}^n;J_0),\, b_\varphi-b_-\big).$$

Proof. It suffices to find an almost complex complex structure $J\in\mathcal{J}$ such that the symplectic area of each non-constant J-holomorphic map $u\colon (D,\partial D)\to (M,\varphi(L))$ is at least $\min\big(\sigma(L,\mathbb{C}^n;J_0), b_\varphi-b_-\big)$. We construct such a J as follows. Transferring the

complex structure J_0 by means of the chart φ, we obtain a complex structure J_0^φ on B_φ. We claim that J_0^φ extends to an almost complex structure $J \in \mathcal{J}$ on M. Indeed, pick an arbitrary $J_1 \in \mathcal{J}$. For each $x \in M$, the space of complex structures $J_{(x)}$ on the tangent space T_xM satisfying $\omega(\xi, J_{(x)}\xi) > 0$ for all $\xi \in T_xM \setminus \{0\}$ is contractible [18]. Thus there is an almost complex structure J on M that coincides with J_0^φ on B_φ, and with J_1 outside a relatively compact neighbourhood of B_φ. Then $J \in \mathcal{J}$.

Let $u\colon (D, \partial D) \to (M, \varphi(L))$ be a non-constant J-holomorphic map. If the image of u is contained in B_φ, then $u_\varphi = \varphi^{-1} \circ u\colon (D, \partial D) \to (\mathbb{C}^n, L)$ is a non-constant holomorphic map. Hence $\int_D u^*\omega = \int_D u_\varphi^*\omega_n \geq \sigma(L, \mathbb{C}^n; J_0)$.

If the image of u is not contained in B_φ, then the set $V_\varphi = \varphi^{-1}(u(D))$ is a real analytic subvariety in $B(b_\varphi)$ intersecting the sphere $\partial B(b_-)$. Applying Theorem A.1 from Appendix A (with $b_- = \pi r_-^2$, $b_\varphi = \pi r_+^2$), we infer that the Riemannian area of V_φ is at least $b_\varphi - b_-$. Since the Riemannian area of a holomorphic curve in $\mathbb{C}^n$ equals its symplectic area, and the symplectic area of u is not less than the symplectic area of V_φ, it follows that the symplectic area of u is at least $b_\varphi - b_-$. □

We claim that $\sigma(T(\boldsymbol{a}), \mathbb{C}^n; J_0) \geq \underline{\boldsymbol{a}}$. Let $u\colon (D, \partial D) \to (\mathbb{C}^n, T(\boldsymbol{a}))$ be a non-constant holomorphic map. Write $u = (u_1, \ldots, u_n)$, where each $u_j\colon (D, \partial D) \to (\mathbb{C}, T(a_j))$ is a holomorphic map. The symplectic area of u is positive, and it equals the sum of the symplectic areas of the maps u_j. Since the symplectic area of u_j is a non-negative integer multiple of a_j, the symplectic area of u is at least $\underline{\boldsymbol{a}}$. The torus $T(\boldsymbol{a})$ is contained in the ball $B^{2n}(|\boldsymbol{a}|)$. By Lemma 2.2, $\sigma(T_\varphi(\boldsymbol{a}), M) \geq \|\boldsymbol{a}\| - |\boldsymbol{a}| = \underline{\boldsymbol{a}}$. In view of (1), we conclude that $e(T_\varphi(\boldsymbol{a}), M) \geq \underline{\boldsymbol{a}}$. This completes the proof of Proposition 2.1. □

2.2. **Deformations.** Let (M, ω) be a symplectic manifold. Denote by $\mathcal{L}$ the space of closed embedded Lagrangian submanifolds in (M, ω) endowed with the C^∞-topology. Given a $\mathrm{Ham}(M, \omega)$-invariant function f on $\mathcal{L}$ taking values in a set X, we associate with each $L \in \mathcal{L}$ a function germ $S_L^f\colon H^1(L; \mathbb{R}) \to X$ at the point $0 \in H^1(L; \mathbb{R})$ following [3]. This construction provides additional invariants of Lagrangian submanifolds. We use it to prove Theorem 1.3 (ii).

By Weinstein's Lagrangian Neighbourhood Theorem, there is a symplectomorphism g from a neighbourhood of L in M to a fibrewise convex neighbourhood of the zero section of T^*L such that the image of L is the zero section [29]. There is a neighbourhood V of the point L in the space $\mathcal{L}$ such that each $L' \in V$ is mapped to the graph of a closed 1-form $\alpha_{L'}$ on L. Consider the mapping $w_{L,V}\colon V \to H^1(L; \mathbb{R})$ that sends $L' \in V$ to the cohomology class of the form $\alpha_{L'}$. This mapping is locally surjective at L. Denote by w_L the germ of $w_{L,V}$ at L. If two Lagrangian submanifolds $L_0, L_1 \in V$ are mapped by $w_{L,V}$ to the same cohomology class $\zeta \in H^1(L; \mathbb{R})$, then they are Hamiltonian isotopic. Indeed, consider the family of Lagrangian submanifolds $\{L_t\}$ such that $g(L_t)$ is the graph of the 1-form $\alpha_t = t\alpha_{L_1} + (1-t)\alpha_{L_0}$ for each $t \in [0, 1]$. Since $[\alpha_t] = \zeta$ for all t, the family $\{L_t\}$ is a Hamiltonian isotopy between L_0 and L_1. Therefore, one can define a mapping germ $S_L^f\colon H^1(L; \mathbb{R}) \to X$ at the point $0 \in H^1(L; \mathbb{R})$ by $S_L^f(\zeta) = f(L')$, where

$w_L(L') = \zeta$. In order to prove that the definition of S_L^f does not depend on the choice of the symplectomorphism g, it suffices to give a description of the mapping germ w_L that does not use g. This description goes as follows: the evaluation of $w_L(L')$ on a 1-homology class $\lambda \in H_1(L;\mathbb{Z})$ equals $\int_{[0,1]\times S^1} h^*\omega$, where $h\colon [0,1]\times S^1 \to M$ is a smooth map with image in a tubular neighbourhood of L such that $h(\{0\}\times S^1)$ is a loop in L representing the class λ and $h(\{1\}\times S^1) \subset L'$.

It immediately follows from the definition that S_L^f is $\mathrm{Ham}(M,\omega)$-invariant in the following sense: for each $\psi \in \mathrm{Ham}(M,\omega)$, we have

$$S^f_{\psi(L)} \;=\; S^f_L \circ (\psi|_L)^* \,, \tag{2}$$

and if, moreover, f is $\mathrm{Symp}(M,\omega)$-invariant, then (2) holds for each $\psi \in \mathrm{Symp}(M,\omega)$. The displacement energy function $e\,(L) = e\,(L,M)$ takes values in $[0,\infty[\,\cup\,\{\infty\}$ and is $\mathrm{Symp}(M,\omega)$-invariant.

Proposition 2.3. *Let $L = T_\varphi(\boldsymbol{a})$ be a product Lagrangian torus in a tame symplectic manifold. Assume that $\|\boldsymbol{a}\| \le b_\varphi$. Then*

$$S^e_L(\zeta) \;=\; e(L) + \min\bigl(l_1(\zeta),\dots,l_{m(\boldsymbol{a})}(\zeta)\bigr),$$

where $l_1,\dots,l_{m(\boldsymbol{a})}$ are independent linear functions on $H^1(L;\mathbb{R})$.

Proof. Consider the mapping germ $\theta\colon (\mathbb{R}^n,0) \to (\mathcal{L},L)$, $\boldsymbol{s} \mapsto T_\varphi(\boldsymbol{a}+\boldsymbol{s})$. The composition $A = w_L \circ \theta\colon (\mathbb{R}^n,0) \to (H^1(L;\mathbb{R}),0)$ is a linear isomorphism germ. Choose $\varepsilon > 0$ so small that $\varphi\colon B^{2n}(b_\varphi+\varepsilon) \to M$ is defined. For $\boldsymbol{s}$ small enough, we have $\|\boldsymbol{a}+\boldsymbol{s}\| \le b_\varphi + \varepsilon$ and hence, by Proposition 2.1,

$$e\,(T_\varphi(\boldsymbol{a}+\boldsymbol{s})) \;=\; \min\bigl(a_1+s_1,\dots,a_n+s_n\bigr). \tag{3}$$

We can assume, after reordering the coordinates, that

$$\underline{\boldsymbol{a}} = a_1 = \cdots = a_{m(\boldsymbol{a})} < a_{m(\boldsymbol{a})+1} \le \cdots \le a_n.$$

For $\boldsymbol{s}$ sufficiently small (say, such that the absolute values of all its components do not exceed $\frac{1}{2}(a_{m(\boldsymbol{a})+1} - a_{m(\boldsymbol{a})})$), in view of (3) we have

$$e\,(T_\varphi(\boldsymbol{a}+\boldsymbol{s})) \;=\; \underline{\boldsymbol{a}} + \min\bigl(s_1,\dots,s_{m(\boldsymbol{a})}\bigr) \;=\; e(L) + \min\bigl(\pi_1(\boldsymbol{s}),\dots,\pi_{m(\boldsymbol{a})}(\boldsymbol{s})\bigr), \tag{4}$$

where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the projection onto the i-th coordinate axis, $\pi_i(\boldsymbol{s}) = s_i$. Since $S^e_L(\zeta) = e\bigl(T_\varphi(\boldsymbol{a} + A^{-1}(\zeta))\bigr)$, it follows from (4) that

$$S^e_L(\zeta) \;=\; e(L) + \min\bigl(l_1(\zeta),\dots,l_{m(\boldsymbol{a})}(\zeta)\bigr)$$

where $l_1 = \pi_1 \circ A^{-1},\dots,l_{m(\boldsymbol{a})} = \pi_{m(\boldsymbol{a})} \circ A^{-1}$ are independent linear functions on $H^1(L;\mathbb{R})$. □

Proof of Theorem 1.3 (ii). Denote $L = T_\varphi(\boldsymbol{a})$, $L' = T_{\varphi'}(\boldsymbol{a}')$. It follows from Proposition 2.1 and symplectic invariance of displacement energy that

$$\underline{\boldsymbol{a}} \;=\; e\,(L,M) \;=\; e\,(L',M) \;=\; \underline{\boldsymbol{a}}'.$$

According to Proposition 2.3, the cohomology classes $\zeta \in H^1(L;\mathbb{R})$ such that $S^e_L(\zeta) = \underline{\boldsymbol{a}}$ form a vector space germ W of dimension $n - m(\boldsymbol{a})$, and the cohomology classes $\zeta' \in H^1(L';\mathbb{R})$ such that $S^e_{L'}(\zeta') = \underline{\boldsymbol{a}}$ form a vector space germ W' of dimension $n - m(\boldsymbol{a}')$. If $L' = \psi(L)$ for some $\psi \in \mathrm{Symp}(M,\omega)$, then $S^e_{L'} = S^e_L \circ A_\psi$, where $A_\psi = (\psi|_L)^*$ is a linear isomorphism between $H^1(L;\mathbb{R})$ and $H^1(L';\mathbb{R})$. Hence $A_\psi(W) = W'$, and therefore $m(\boldsymbol{a}) = m(\boldsymbol{a}')$. □

2.3. **Symplectic area class and Maslov class.** Given a Lagrangian submanifold L of a symplectic manifold (M,ω), one can consider two relative cohomology classes: the symplectic area class $\sigma_L \in H^2(M,L;\mathbb{R})$ represented by the 2-form ω, and the Maslov class $\mu_L \in H^2(M,L;\mathbb{Z})$, defined as in [27]. Both σ and μ are symplectically invariant in the sense that $\sigma_{\psi(L)} = \psi^*\sigma_L$ and $\mu_{\psi(L)} = \psi^*\mu_L$ for each symplectomorphism ψ. These classes define homomorphisms from $\pi_2(M,L)$ to $\mathbb{R}$ that we shall also denote by σ_L and μ_L. Define the subgroup $\Gamma(L) \subset \mathbb{R}$ to be the image of the subgroup $\ker(\mu_L) \subset \pi_2(M,L)$ under the homomorphism $\sigma_L \colon \pi_2(M,L) \to \mathbb{R}$. Since σ_L and μ_L are symplectically invariant, so is $\Gamma(L)$:

Lemma 2.4. *Let L, L' be Lagrangian submanifolds of (M,ω). If $L \sim L'$, then $\Gamma(L) = \Gamma(L')$.*

Theorem 1.3 (i) is a corollary of Lemma 2.4 and the following assertion:

Lemma 2.5. *Let $T_\varphi(\boldsymbol{a})$ be a product Lagrangian torus in a symplectically aspherical symplectic manifold (M,ω). Then $\Gamma\big(T_\varphi(\boldsymbol{a})\big) = \Gamma(\boldsymbol{a})$.*

Proof. For $i \in \{1,\dots,n\}$, let D_i be a disc in $\mathbb{R}^{2n}$ with boundary on $T(\boldsymbol{a})$ such that the projection of D^i to the i-th factor in $\mathbb{R}^2 \times \dots \times \mathbb{R}^2 = \mathbb{R}^{2n}$ is the disc in $\mathbb{R}^2$ bounded by the circle $T(a_i)$, and the projections to other factors are points. Denote by $\hat{D}_i$ the element of $\pi_2\left(\mathbb{R}^{2n}, T(\boldsymbol{a})\right)$ represented by D_i. The classes $\hat{D}_1,\dots,\hat{D}_n$ generate the free Abelian group $\pi_2\left(\mathbb{R}^{2n}, T(\boldsymbol{a})\right)$. Denote $\tilde{D}_i = \varphi_*\hat{D}_i \in \pi_2(M,L)$ where $L := T_\varphi(\boldsymbol{a})$. For each i, we have $\sigma_{T(\boldsymbol{a})}(\hat{D}_i) = a_i$, $\mu_{T(\boldsymbol{a})}(\hat{D}_i) = 2$, and hence $\sigma_L(\tilde{D}_i) = a_i$, $\mu_L(\tilde{D}_i) = 2$. The group $\pi_2(M,L)$ is the direct sum of $\pi_2(M)$ and the subgroup generated by the elements $\tilde{D}_i$. Since (M,ω) is symplectically aspherical and $\mu_L|_{H_2(M;\mathbb{Z})} = 2c_1(\omega)$ (see [27]), both σ_L and μ_L vanish on $\pi_2(M)$. The kernel of μ_L is the direct sum of $\pi_2(M)$ and the subgroup generated by the differences $\tilde{D}_i - \tilde{D}_j$, where $i,j \in \{1,\dots,n\}$ and j is such that $\underline{\boldsymbol{a}} = a_j$. Therefore, $\sigma_L(\ker\mu_L)$ consists of all integer combinations of the numbers $a_i - \underline{\boldsymbol{a}} = \sigma(\tilde{D}_i - \tilde{D}_j)$. □

3. Proof of Theorem 1.2

3.1. **Generalized Clifford tori in $\mathbb{CP}^n$.** We consider a certain class of product Lagrangian tori in the complex projective space, the so-called *generalized Clifford tori.* Identify the symplectic space $(\mathbb{R}^{2n},\omega_n)$ with $\mathbb{C}^n$, the complex coordinates being $z_1 = x_1 + iy_1,\dots,z_n = x_n + iy_n$. Consider the diagonal action of the Lie group $U(1)$ on the space $\mathbb{C}^n$.

For each $b > 0$, the sphere $S^{2n-1}(b) = \partial B^{2n}(b)$ is invariant under this action. Denote by $\mathbb{C}P^{n-1}(b)$ the quotient $S^{2n-1}(b)/U(1)$. The restriction of the symplectic form ω_n to $S^{2n-1}(b)$ is the pullback of a certain symplectic form ω^b_{n-1} on $\mathbb{C}P^{n-1}(b)$. This form is a multiple of the Fubini–Study form.

If $\boldsymbol{a} \in \mathbb{R}^n_+$ and $|\boldsymbol{a}| = b$, then the torus $T(\boldsymbol{a})$ is contained in the sphere $S^{2n-1}(b)$. Moreover, $T(\boldsymbol{a})$ is invariant under the action of $U(1)$. Therefore, the quotient $\widehat{T}(\boldsymbol{a}) = T(\boldsymbol{a})/U(1)$ is a Lagrangian $(n-1)$-torus in $\mathbb{C}P^{n-1}(b)$. It is called a generalized Clifford torus.

Denote by $Z_n(b)$ the complex hypersurface

$$\big(S^{2n-1}(b) \cap \{z_n = 0\}\big)/U(1) \cong \mathbb{C}P^{n-2}$$

in $\mathbb{C}P^{n-1}(b)$, and by $\mathring{B}^{2n-2}(b)$ the open ball $\mathrm{Int}(B^{2n-2}(b))$. The tori $\widehat{T}(\boldsymbol{a})$ are product tori:

Proposition 3.1. *There is a symplectomorphism*

$$\varphi^b_{n-1} \colon \big(\mathring{B}^{2n-2}(b), \omega_{n-1}\big) \to \big(\mathbb{C}P^{n-1}(b) \setminus Z_n(b), \omega^b_{n-1}\big)$$

that maps each product torus $T(a_1, \dots, a_{n-1})$ contained in $\mathring{B}^{2n-2}(b)$ to the torus $\widehat{T}(a_1, \dots, a_n)$, where $a_n = b - a_1 - \cdots - a_{n-1}$.

Proof. Denote by W the subset of $S^{2n-1}(b)$ formed by points with z_n coordinate positive real. Consider the projection of $\mathbb{C}^n$ onto $\mathbb{C}^{n-1}$ defined by forgetting the last coordinate. Restricting this projection to W be obtain a diffeomorphism $\psi \colon W \to \mathring{B}^{2n-2}(b)$. We claim that ψ is a symplectomorphism from $(W, \omega_n|_W)$ onto $\big(\mathring{B}^{2n-2}(b), \omega_{n-1}\big)$. This statement is equivalent to the assertion that the restriction of the 2-form $dx_n \wedge dy_n$ to W vanishes. The latter follows since y_n vanishes on W.

The manifold $S^{2n-1}(b) \setminus \{z_n = 0\}$ is foliated by the orbits of the $U(1)$-action. Each of these orbits intersects W exactly once, and the intersection is transverse. Therefore, symplectic reduction gives rise to a canonical symplectomorphism ψ' from $(W, \omega_n|_W)$ onto $\big(\mathbb{C}P^{n-1}(b) \setminus Z_n(b), \omega^b_{n-1}\big)$.

The composition $\varphi^b_{n-1} = \psi' \circ \psi^{-1}$ is the required symplectomorphism. To prove the assertion concerning Lagrangian tori, it suffices to observe that the image of $T(a_1, \dots, a_{n-1})$ under the symplectomorphism ψ^{-1} is the torus $T(a_1, \dots, a_{n-1}) \times \sqrt{a_n/\pi}$, and that the $U(1)$-orbits passing through the latter torus form the torus $T(a_1, \dots, a_n)$. □

Proposition 3.2. *Let $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{R}^n_+$ be such that $|\boldsymbol{a}| = |\boldsymbol{a}'|$. Consider the Lagrangian tori $\widehat{T}(\boldsymbol{a})$, $\widehat{T}(\boldsymbol{a}')$ in $\mathbb{C}P^{n-1}(|\boldsymbol{a}|)$. If $\widehat{T}(\boldsymbol{a}) \sim \widehat{T}(\boldsymbol{a}')$, then $\boldsymbol{a} \simeq \boldsymbol{a}'$.*

Proof. By Theorem 1.3 (ii) we have $\underline{\boldsymbol{a}} = \underline{\boldsymbol{a}}'$ and $m(\boldsymbol{a}) = m(\boldsymbol{a}')$. In view of Lemma 2.4, it remains to show that $\Gamma(\widehat{T}(\boldsymbol{a})) = \Gamma(\boldsymbol{a})$. Let $\hat{D}_1, \dots, \hat{D}_{n-1}$ be the elements of the group $\pi_2\left(\mathbb{R}^{2n-2}, T(a_1, \dots, a_{n-1})\right)$ defined as in the proof of Lemma 2.5. The symplectomorphism $\varphi^{|\boldsymbol{a}|}_{n-1}$ sends these classes to the classes $\tilde{D}_1, \dots, \tilde{D}_{n-1}$ in $\pi_2(\mathbb{C}P^{n-1}(|\boldsymbol{a}|), \widehat{T}(\boldsymbol{a}))$. For each i, we have $\sigma_{\widehat{T}(\boldsymbol{a})}(\tilde{D}_i) = a_i$, $\mu_{\widehat{T}(\boldsymbol{a})}(\tilde{D}_i) = 2$. The free Abelian group $\pi_2(\mathbb{C}P^{n-1}(|\boldsymbol{a}|), \widehat{T}(\boldsymbol{a}))$ is generated by the classes $\tilde{D}_1, \dots, \tilde{D}_{n-1}$, and the class $[\mathbb{C}P^1]$ represented by a complex line in the complex projective space.

We have $\mu_{\widehat{T}(\boldsymbol{a})}\big([\mathbb{C}P^1]\big) = 2n$, since the value of the Maslov class on $\mathbb{C}P^1$ is twice the value of $c_1(T\mathbb{C}P^{n-1})$. We claim that $\sigma_{\widehat{T}(\boldsymbol{a})}\big([\mathbb{C}P^1]\big) = |\boldsymbol{a}|$. Indeed, let $\mathbb{C}P^1 \subset \mathbb{C}\mathrm{P}^{n-1}$ be the quotient of the sphere $\{z_2 = \cdots = z_{n-1} = 0\} \cap S^{2n-1}(|\boldsymbol{a}|)$ by the diagonal action of $U(1)$. The symplectomorphism $\varphi_{n-1}^{|\boldsymbol{a}|}$ identifies the complement of a point in $\mathbb{C}P^1$ with the open symplectic disc $\mathring{B}^{2n-2}(|\boldsymbol{a}|) \cap \{z_2 = \cdots = z_{n-1} = 0\}$. This disc has area $|\boldsymbol{a}|$, and hence the integral of the symplectic form $\omega_{n-1}^{|\boldsymbol{a}|}$ over $\mathbb{C}P^1$ equals $|\boldsymbol{a}|$.

Define $\tilde{D}_n = [\mathbb{C}P^1] - \sum_{i=1}^{n-1} \tilde{D}_i$. The group $\pi_2(\mathbb{C}P^{n-1}(|\boldsymbol{a}|), \widehat{T}(\boldsymbol{a}))$ is generated by the classes $\tilde{D}_1, \dots, \tilde{D}_n$, and we have $\sigma_{\widehat{T}(\boldsymbol{a})}(\tilde{D}_n) = a_n$, $\mu_{\widehat{T}(\boldsymbol{a})}(\tilde{D}_n) = 2$. The kernel of $\mu_{\widehat{T}(\boldsymbol{a})}$ is generated by the differences $\tilde{D}_i - \tilde{D}_j$, where $i, j \in \{1, \dots, n\}$ and j is such that $\underline{\boldsymbol{a}} = a_j$. Therefore, $\sigma_{\widehat{T}(\boldsymbol{a})}(\ker \mu_{\widehat{T}(\boldsymbol{a})})$ consists of all integer combinations of the numbers $a_i - \underline{\boldsymbol{a}} = \sigma(\tilde{D}_i - \tilde{D}_j)$. □

3.2. **Proof of Theorem 1.2.** Arguing by contradiction, we suppose that $T(\boldsymbol{a}) \approx T(\boldsymbol{a}')$ in $B^{2n}(b)$. By Theorem 1.3 (ii), with (M, ω) a large ball and φ, φ' the identity embeddings, we have $\underline{\boldsymbol{a}} = \underline{\boldsymbol{a}}'$. We can assume that $\|\boldsymbol{a}\| \geq \|\boldsymbol{a}'\|$. Since $\underline{\boldsymbol{a}} = \underline{\boldsymbol{a}}'$ and, by hypothesis, $|\boldsymbol{a}| \neq |\boldsymbol{a}'|$, we have $\|\boldsymbol{a}\| - \|\boldsymbol{a}'\| = |\boldsymbol{a}| - |\boldsymbol{a}'| > 0$. By hypothesis we have $|\boldsymbol{a}| \leq b < \|\boldsymbol{a}\|$. Thus $|\boldsymbol{a}'| < |\boldsymbol{a}| \leq b < |\boldsymbol{a}| + \underline{\boldsymbol{a}}$. Choose $c' < c$ such that

$$b < c' < c < |\boldsymbol{a}| + \underline{\boldsymbol{a}}.$$

Define $a_{n+1} := c - |\boldsymbol{a}|$ and $a'_{n+1} := c - |\boldsymbol{a}'|$. Then $a_{n+1} < a'_{n+1}$ and $a_{n+1} = c - |\boldsymbol{a}| < \underline{\boldsymbol{a}}$. Therefore,

$$\min\{a_1, \dots, a_n, a_{n+1}\} \;=\; a_{n+1} < \min\{\underline{\boldsymbol{a}}, a'_{n+1}\} \;=\; \min\{a'_1, \dots, a'_n, a'_{n+1}\}. \tag{5}$$

Recall that $T(\boldsymbol{a}) \approx T(\boldsymbol{a}')$ in $B^{2n}(b)$. Cutting off the Hamiltonian function that generates this isotopy, we construct a Hamiltonian isotopy supported in $B^{2n}(c')$ that moves $T(\boldsymbol{a})$ to $T(\boldsymbol{a}')$. The symplectomorphism φ_n^c from Proposition 3.1 transfers this isotopy to a Hamiltonian isotopy of $\mathbb{C}\mathrm{P}^n(c)$. It moves $\widehat{T}(a_1, \dots, a_n, a_{n+1})$ to $\widehat{T}(a'_1, \dots, a'_n, a'_{n+1})$. By Proposition 3.2, $\min\{a_1, \dots, a_n, a_{n+1}\} = \min\{a'_1, \dots, a'_n, a'_{n+1}\}$, in contradiction to (5). □

4. Constructions of Hamiltonian isotopies

4.1. **Proof of Theorem 1.1 (i).** The unitary group $U(n)$ acts on $\mathbb{C}^n$ preserving the symplectic form ω_n. Since a permutation of coordinates $z_1, \dots, z_n$ is a unitary map and the group $U(n)$ is path-connected, there is a smooth family $\{\Phi_t\}$, $t \in [0,1]$, of unitary maps such that $\Phi_0 = \mathtt{id}$ and $\Phi_1(T(\boldsymbol{a})) = T(\boldsymbol{a}')$. The flow $\{\Phi_t\}$ is Hamiltonian because $\mathbb{C}^n$ is simply-connected. □

4.2. The proof of Theorem 1.1 (ii) relies on the following lemma, which represents a special case of Theorem 1.1 (ii).

Lemma 4.1. *For each positive a, c, and d, the tori $T(a, a+c, a+d)$ and $T(a, a+c+d, a+d)$ are Hamiltonian isotopic in the ball $B^6(4a+c+2d)$.*

Proof. Let $W = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2|\}$. Consider the map

$$\Psi \colon W \to \mathbb{C}^2, \quad (z_1, z_2) \mapsto \left(\frac{z_1 z_2}{|z_2|}, \frac{z_2\sqrt{|z_2|^2 - |z_1|^2}}{|z_2|} \right).$$

It is injective, and its image is the complement of the complex line $\{z_2 = 0\}$. We claim that Ψ preserves the symplectic form $\omega_2 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Indeed, write $z_1 = e^{2\pi i \theta_1}\sqrt{\rho_1/\pi}$, $z_2 = e^{2\pi i \theta_2}\sqrt{\rho_2/\pi}$, with θ_1, θ_2 in $S^1 = \mathbb{R}/\mathbb{Z}$ and ρ_1, ρ_2 nonnegative real. For nonzero values of z_2, we have $\omega_2 = d\rho_1 \wedge d\theta_1 + d\rho_2 \wedge d\theta_2$ and

$$\Psi(\rho_1, \theta_1, \rho_2, \theta_2) = (\rho_1, \theta_1 + \theta_2, \rho_2 - \rho_1, \theta_2).$$

Clearly, Ψ is symplectic outside the complex line $\{z_2 = 0\}$, and hence, by continuity, on the whole of W. A product torus $T(a_0, a_0+b_0) \subset W$ is mapped by Ψ to the torus $T(a_0, b_0)$.

The torus $T(a, a+c+d, a+d)$ is Hamiltonian isotopic, through a unitary isotopy, to the torus $T(a+d, a+c+d, a)$ in the ball $B^6(3a+c+2d)$. Therefore, it suffices to prove that the tori $T(a, a+c, a+d)$ and $T(a+d, a+c+d, a)$ are Hamiltonian isotopic in $B^6(4a+c+2d)$.

Consider the map $\Psi_+ = \Psi \times \mathtt{id} \colon W \times \mathbb{C} \to \mathbb{C}^3$. We have $\Psi_+(T(a, a+c, a+d)) = T(a, c, a+d)$ and $\Psi_+(T(a+d, a+c+d, a)) = T(a+d, c, a)$. The Hamiltonian function $H = \frac{\pi}{2}(x_1 y_3 - x_3 y_1)$ gives rise to a unitary Hamiltonian flow $\{\Phi_t\}$ that does not change the complex coordinate z_2. We have $\Phi_1(z_1, z_2, z_3) = (z_3, z_2, -z_1)$. In particular, Φ_1 maps $T(a, c, a+d)$ to $T(a+d, c, a)$. Multiplying H by an appropriate cutoff function, we construct a Hamiltonian H', compactly supported in $\mathbb{C}^3 \setminus \{z_2 = 0\}$, whose flow $\{\Phi'_t\}$ moves the torus $T(a, c, a+d)$ in exactly the same way as the flow $\{\Phi_t\}$. Consider the Hamiltonian flow $\{\Phi_t^+\}$ on $\mathbb{C}^3$ generated by the Hamiltonian function $H' \circ \Psi_+$. This flow is compactly supported in $W \times \mathbb{C}$, where $\Phi_t^+ = \Psi_+^{-1} \circ \Phi'_t \circ \Psi_+$. In particular, $\Phi_t^+(T(a, a+c, a+d)) = \Psi_+^{-1}(\Phi_t(T(a, c, a+d)))$ for all values of t, and $\Phi_1^+(T(a, a+c, a+d)) = T(a+d, a+c+d, a)$. It remains to show that each torus $\Phi_t^+(T(a, a+c, a+d))$ is contained in $B^6(4a+c+2d)$.

Let $(z_1, z_2, z_3) \in \Phi_t^+(T(a, a+c, a+d))$. We are to prove that $\pi(|z_1|^2 + |z_2|^2 + |z_3|^2) \leq 4a + c + 2d$. The point $\Psi_+(z_1, z_2, z_3) = (z'_1, z'_2, z_3)$ belongs to the torus $\Phi_t(T(a, c, a+d))$. Since $T(a, c, a+d)$ is contained in the sphere $\partial B^6(2a+c+d)$ and Φ_t is unitary, it follows that $(z'_1, z'_2, z_3) \in \partial B^6(2a+c+d)$. Hence $\pi(|z'_1|^2 + |z'_2|^2 + |z_3|^2) = 2a + c + d$. By the construction of Φ_t, we have $\pi|z'_2|^2 = c$. The definition of the map Ψ_+ implies that $|z'_1| = |z_1|$ and $|z_2|^2 = |z'_1|^2 + |z'_2|^2$. Therefore,

$$\begin{aligned} \pi(|z_1|^2 + |z_2|^2 + |z_3|^2) &= 2a + c + d + \pi|z'_1|^2 \\ &= 4a + 2c + 2d - \pi|z'_2|^2 - \pi|z_3|^2 \\ &= 4a + c + 2d - \pi|z_3|^2 \leq 4a + c + 2d, \end{aligned}$$

as we wished to show. $\square$

Lemma 4.2. *Let $\boldsymbol{c} = (c_1, \dots, c_k)$ and $\boldsymbol{c}' = (c'_1, \dots, c'_k)$ be vectors in $\mathbb{R}^k_+$, $k \geq 2$, such that, for some different indices $i, j \in \{1, \dots, k\}$, we have $c'_i = c_i + c_j$, and $c'_l = c_l$ for $l \neq i$. For each $n > k$ and each positive a, the n-dimensional tori $T(\boldsymbol{p}) = T(a, \dots, a, a+c_1, \dots, a+c_k)$ and $T(\boldsymbol{p}') = T(a, \dots, a, a + c'_1, \dots, a + c'_k)$ are Hamiltonian isotopic in the ball $B^{2n}(\|\boldsymbol{p}'\|)$.*

Proof. We may assume that $i = 1$ and $j = 2$ after applying to the tori $T(\boldsymbol{p})$ and $T(\boldsymbol{p}')$ unitary isotopies that swap the complex coordinates z_{n-k+1} and z_{n-k+i}, z_{n-k+2} and z_{n-k+j}. By Lemma 4.1, there is a Hamiltonian isotopy on $\mathbb{C}^3$ that moves the torus $L_0 = T(a, a + c_1, a + c_2)$ to $L_1 = T(a, a + c_1 + c_2, a + c_2)$ through tori L_t belonging to $B^6(4a + c_1 + 2c_2)$. The tori $L'_0 = T(\boldsymbol{p})$ and $L'_1 = T(\boldsymbol{p}')$ are Hamiltonian isotopic through the family $L'_t = T(a, \dots, a) \times L_t \times T(a + c_3, \dots, a + c_k)$. All the tori L'_t are contained in the ball

$$B^{2n}\big((n + 1)a + |\boldsymbol{c}| + c_2\big) = B^{2n}\big(\|\boldsymbol{p}'\|\big).$$

$\square$

4.3. **Proof of Theorem 1.1 (ii).** After applying appropriate unitary isotopies to the tori $T(\boldsymbol{a})$ and $T(\boldsymbol{a}')$, we may assume that the first $m(\boldsymbol{a})$ components of both $\boldsymbol{a}$ and $\boldsymbol{a}'$ equal $\underline{\boldsymbol{a}}$. Let $k = n - m(\boldsymbol{a})$. Write

$$T(\boldsymbol{a}) = T(\underline{\boldsymbol{a}}, \dots, \underline{\boldsymbol{a}}, \underline{\boldsymbol{a}} + d_1, \dots, \underline{\boldsymbol{a}} + d_k), \quad T(\boldsymbol{a}') = T(\underline{\boldsymbol{a}}, \dots, \underline{\boldsymbol{a}}, \underline{\boldsymbol{a}} + e_1, \dots, \underline{\boldsymbol{a}} + e_k),$$

where $\boldsymbol{d} = (d_1, \dots, d_k)$ and $\boldsymbol{e} = (e_1, \dots, e_k)$ are vectors in $\mathbb{R}^k_+$. If k equals 1, then the hypothesis $\Gamma(\boldsymbol{a}) = \Gamma(\boldsymbol{a}')$ implies that $\boldsymbol{a} = \boldsymbol{a}'$, and there is nothing to prove. Assume that $k \geq 2$.

Suppose that there is a sequence $\boldsymbol{d} = \boldsymbol{d}^0, \boldsymbol{d}^1, \boldsymbol{d}^2, \dots, \boldsymbol{d}^\ell = \boldsymbol{e}$ of vectors in $\mathbb{R}^k_+$ with the following property: for each $s \in \{1, \dots, \ell\}$, the vector $\boldsymbol{d}^s = (d^s_1, \dots, d^s_k)$ is obtained from the vector $\boldsymbol{d}^{s-1}$ either by swapping two of the components, or by adding to the i-th component the j-th component, or by subtracting from the i-th component the j-th component. For $s \in \{0, \dots, \ell\}$, define $\boldsymbol{a}^s = (\underline{\boldsymbol{a}}, \dots, \underline{\boldsymbol{a}}, \underline{\boldsymbol{a}} + d^s_1, \dots, \underline{\boldsymbol{a}} + d^s_k)$. Consider the sequence of tori $T(\boldsymbol{a}) = T(\boldsymbol{a}^0), T(\boldsymbol{a}^1), \dots, T(\boldsymbol{a}^\ell) = T(\boldsymbol{a}')$. For each $s \in \{1, \dots, \ell\}$, the tori $T(\boldsymbol{a}^{s-1})$ and $T(\boldsymbol{a}^s)$ are Hamiltonian isotopic inside the ball $B^{2n}\big(\max(\|\boldsymbol{a}^{s-1}\|, \|\boldsymbol{a}^s\|)\big)$. Indeed, if $\boldsymbol{d}^{s-1}$ and $\boldsymbol{d}^s$ are related by a swap of components, then there is a unitary isotopy; otherwise, we apply Lemma 4.2 with either $\boldsymbol{c} = \boldsymbol{d}^{s-1}$, $\boldsymbol{c}' = \boldsymbol{d}^s$, or $\boldsymbol{c} = \boldsymbol{d}^s$, $\boldsymbol{c}' = \boldsymbol{d}^{s-1}$.

It thus suffices to show that such a sequence $\boldsymbol{d}^s$ indeed exists and that, moreover, we have $\|\boldsymbol{a}^s\| \leq \|\boldsymbol{a}\|$ or $\|\boldsymbol{a}^s\| \leq \|\boldsymbol{a}'\|$ for all s. To this end, we apply Theorem B.1 from Appendix B (we refer the reader to the definitions therein). The theorem is applicable because the condition $\Gamma(\boldsymbol{a}) = \Gamma(\boldsymbol{a}')$ means exactly $\langle \boldsymbol{d} \rangle = \langle \boldsymbol{e} \rangle$. Consider the sequence (path) $\boldsymbol{d} = \boldsymbol{d}^0, \boldsymbol{d}^1, \boldsymbol{d}^2, \dots, \boldsymbol{d}^\ell = \boldsymbol{e}$ whose existence is guaranteed by Theorem B.1. Since it is low, it follows that, for each s, we have $|\boldsymbol{d}^s| \leq |\boldsymbol{d}|$ or $|\boldsymbol{d}^s| \leq |\boldsymbol{e}|$, and hence $\|\boldsymbol{a}^s\| \leq \|\boldsymbol{a}\|$ or $\|\boldsymbol{a}^s\| \leq \|\boldsymbol{a}'\|$. Thus, this sequence has the required properties and the proof of Theorem 1.1 is complete. $\square$

5. Spaces of symplectic charts and product tori

Given $b > 0$, denote by $\operatorname{Emb}(B^{2n}(b), M, \omega)$ the space of symplectic charts $\varphi\colon B^{2n}(b) \to (M, \omega)$, endowed with the C^∞-topology. By Darboux's theorem, this space is nonempty at least for sufficiently small b. The *Gromov radius* $\rho(M, \omega)$ of (M, ω) is defined as the supremum of all b such that $\operatorname{Emb}(B^{2n}(b), M, \omega)$ is nonempty (we allow $\rho(M, \omega) = \infty$). For computations and estimates of $\rho(M, \omega)$ we refer to [23] and the references therein. It has been conjectured that the space $\operatorname{Emb}(B^{2n}(b), M, \omega)$ is connected for all closed symplectic manifolds and all $b > 0$. This has been proved for certain closed 4-manifolds and also for the symplectic 4-ball $\mathring{B}(c)$, see [17].

Theorem 5.1. *Let $T_\varphi(\boldsymbol{a})$ and $T_{\varphi'}(\boldsymbol{a}')$ be two Lagrangian product tori in a symplectically aspherical tame symplectic manifold (M, ω).*

(i) *Let $b_- = \min\{b_\varphi, b_{\varphi'}\}$ and $b_+ = \max\{b_\varphi, b_{\varphi'}\}$. Assume that the space $\operatorname{Emb}(B^{2n}(b_-), M, \omega)$ is path-connected and that $\max\{\|\boldsymbol{a}\|, \|\boldsymbol{a}'\|\} \le b_+$. Then the conditions $\boldsymbol{a} \simeq \boldsymbol{a}'$, $T_\varphi(\boldsymbol{a}) \sim T_{\varphi'}(\boldsymbol{a}')$, $T_\varphi(\boldsymbol{a}) \approx T_{\varphi'}(\boldsymbol{a}')$ are equivalent one to another.*

(ii) *Assume that the space $\operatorname{Emb}(B^{2n}(b), M, \omega)$ is path connected for all values of b and that $\max\{\|\boldsymbol{a}\|, \|\boldsymbol{a}'\|\} < \rho(M, \omega)$. Then the conditions $\boldsymbol{a} \simeq \boldsymbol{a}'$, $T_\varphi(\boldsymbol{a}) \sim T_{\varphi'}(\boldsymbol{a}')$, $T_\varphi(\boldsymbol{a}) \approx T_{\varphi'}(\boldsymbol{a}')$ are equivalent one to another.*

Proof. First we prove statement (i). If $T_\varphi(\boldsymbol{a}) \approx T_{\varphi'}(\boldsymbol{a}')$, then $T_\varphi(\boldsymbol{a}) \sim T_{\varphi'}(\boldsymbol{a}')$ by definition. If $T_\varphi(\boldsymbol{a}) \sim T_{\varphi'}(\boldsymbol{a}')$, then $\boldsymbol{a} \simeq \boldsymbol{a}'$ by Theorem 1.3. Let $\boldsymbol{a} \simeq \boldsymbol{a}'$. We can assume that $b_- = b_\varphi$ and $b_+ = b_{\varphi'}$. It follows from Theorem 1.1 that $T_{\varphi'}(\boldsymbol{a}) \approx T_{\varphi'}(\boldsymbol{a}')$. Since $\operatorname{Emb}(B^{2n}(b_-), M, \omega)$ is path-connected, there exists a smooth family $\{\varphi_s\}$, $s \in [0,1]$, of symplectic embeddings $B^{2n}(b_-) \to (M, \omega)$ such that $\varphi_0 = \varphi$ and φ_1 coincides with φ' on $B^{2n}(b_-)$. Then there is a Hamiltonian isotopy $\{\Psi_s\}$, $s \in [0,1]$, of (M, ω) such that $\Psi_s \circ \varphi = \varphi_s$ for all s. We have $\Psi_1(T_\varphi(\boldsymbol{a})) = T_{\varphi'}(\boldsymbol{a})$. Therefore, $T_\varphi(\boldsymbol{a}) \approx T_{\varphi'}(\boldsymbol{a}) \approx T_{\varphi'}(\boldsymbol{a}')$.

The statement (ii) will follow from the statement (i) if we show that, for each b, b' satisfying $0 < b < b' < \rho(M, \omega)$, every symplectic embedding $\varphi\colon B^{2n}(b) \to (M, \omega)$ extends to a symplectic embedding $: B^{2n}(b') \to (M, \omega)$. Pick $\varphi_+ \in \operatorname{Emb}(B^{2n}(b'), M, \omega)$. Denote by φ'_+ the restriction of φ_+ to $\varphi\colon B^{2n}(b)$. Since $\operatorname{Emb}(B^{2n}(b), M, \omega)$ is path-connected, we conclude, arguing as above, that there is $\Psi \in \operatorname{Symp}(M, \omega)$ such that $\Psi \circ \varphi = \varphi'_+$. Then $\Psi^{-1} \circ \varphi_+ \in \operatorname{Emb}(B^{2n}(b'), M, \omega)$ is an extension of φ. □

Proposition 5.2. *For a forward complete Liouville manifold (M, ω), the space $\operatorname{Emb}(B^{2n}(b), M, \omega)$ is nonempty and path-connected for each $b > 0$.*

Proof. Let X be a forward complete Liouville field on (M, ω). Denote by $\{f_t\}$, $t \ge 0$, its forward flow. Assume that the space $\operatorname{Emb}(B^{2n}(b), M, \omega)$ is nonempty and pick $\varphi \in (B^{2n}(b), M, \omega)$. Since $(f_t)^* \omega = e^t \omega$ for all $t \ge 0$, the map

$$B^{2n}(e^{2t} b) \to M, \quad x \mapsto f_{2t}(\varphi(e^{-t} x))$$

is a symplectic embedding, and hence the space $\operatorname{Emb}(B^{2n}(b_+), M, \omega)$ is nonempty for all $b_+ > b$.

Let $\varphi, \varphi' \colon B^{2n}(b) \to (M, \omega)$. We prove that φ and φ' are homotopic through symplectic embeddings. After composing φ' with an appropriate Hamiltonian symplectomorphism of (M, ω), we can assume that $\varphi(0) = \varphi'(0)$. Since each element of the linear symplectic group $\mathrm{Sp}(2n; \mathbb{R})$ can be realized as linearization of a Hamiltonian symplectomorphism preserving the point $\varphi(0)$, we can also assume that $d\varphi(0) = d\psi(0)$. There is a symplectic isotopy $\{F_t\}$, $t \in [0,1]$, of $B^{2n}(b)$ such that $F_0 = \mathtt{id}$ and $\psi \circ F_1$ coincides with φ on $B^{2n}(b')$ for some $b' \in\,]0, b[$, see e.g. Appendix A.1 of [10] or the proof of Lemma 2.2 in [22]. Therefore, we may assume that $\varphi = \psi$ on $B^{2n}(b')$.

Consider smooth families $\{\Phi_t\}, \{\Psi_t\}$, $t \geq 0$, of embeddings $B^{2n}(b) \to (M, \omega)$ defined by

$$\Phi_t(x) = (f_{2t} \circ \varphi)\,(e^{-t}x), \quad \Psi_t(x) = (f_{2t} \circ \psi)\,(e^{-t}x).$$

Since $(f_t)^*\,\omega = e^t\omega$, the embeddings Φ_t, Ψ_t are symplectic. Moreover, $\Phi_0 = \varphi$ and $\Psi_0 = \psi$. For $T > 0$ so large that $e^{-T}B^{2n}(b) \subset B^{2n}(b')$, we have $\Phi_T = \Psi_T$. Concatenating the path of embeddings Φ_t, $t \in [0, T]$, from φ to Φ_T with the path of embeddings Ψ_{T-t}, $t \in [0,T]$, from $\Phi_T = \Psi_T$ to ψ, we obtain a required path of symplectic charts from φ to ψ. □

Remark. In the case where (M^{2n}, ω) is a cotangent bundle $(T^*Q, d\lambda)$, a parametric version of the above argument gives a description of the homotopy type of the space $\mathrm{Emb}\,(B^{2n}(b), T^*Q)$: the map $\mathrm{Emb}\,(B^{2n}(b), T^*Q) \to Q$ defined by projecting the center of the ball to the base is a Serre fibration with fibre homotopy equivalent to $U(n)$.

Proof of Theorem 1.4. If we prove that (M, ω) is symplectically aspherical, then the theorem will follow from Proposition 5.2 and Theorem 5.1. Let X be a forward complete Liouville field on (M, ω). Denote by $\{f_t\}$, $t \geq 0$, its forward flow. Let $g \colon S^2 \to M$ be a smooth map. Denote $g_t = f_t \circ g$. Since ω is closed and all maps g_t are homotopic, we have

$$\int_{S^2} g^*\omega \;=\; \int_{S^2} g_t^*\omega \;=\; \int_{S^2} g^*(f_t^*\omega) \;=\; e^t \int_{S^2} g^*\omega$$

for each $t > 0$. Thus $\int_{S^2} g^*\omega$ vanishes, and (M, ω) is symplectically aspherical. □

If the space $\mathrm{Emb}\,(B^{2n}(b), M, \omega)$ is not connected, the classification of product tori can be more complicated:

Example 5.3. The *camel space* with eye of size $c > 0$ is the open subset

$$\mathcal{C}^{2n}(c) \;=\; \{x_1 < 0\} \cup \{x_1 > 0\} \cup \mathring{B}^{2n}(c)$$

of $(\mathbb{R}^{2n}, \omega_n)$. Fix $b > 0$ and define the symplectic embeddings $\varphi_\pm \colon B^{2n}(b) \to \mathcal{C}^{2n}(c)$ by

$$\varphi_\pm(x_1, y_1, \ldots, x_n, y_n) \;=\; \left(x_1 \pm \sqrt{b/\pi}, y_1, \ldots, x_n, y_n\right).$$

If $b \geq c$, then the maps $\varphi_\pm$ are not homotopic through symplectic embeddings, see [6, 20, 28], and hence $\mathrm{Emb}\,(B^{2n}(b), \mathcal{C}^{2n}(c), \omega_n)$ has at least 2 components. Let $\boldsymbol{a} \in \mathbb{R}^{2n}_+$ be such that $T(\boldsymbol{a}) \subset B^{2n}(b)$. The symplectomorphism

$$(x_1, y_1, \ldots, x_n, y_n) \;\mapsto\; (-x_1, -y_1, x_2, y_2, \ldots, x_n, y_n)$$

maps $\varphi_-\big(T(\boldsymbol{a})\big)$ to $\varphi_+\big(T(\boldsymbol{a})\big)$, and hence $\varphi_-\big(T(\boldsymbol{a})\big) \sim \varphi_+\big(T(\boldsymbol{a})\big)$. However, if $\boldsymbol{a}$ is such that $\underline{\boldsymbol{a}} \geq c$, then $\varphi_-\big(T(\boldsymbol{a})\big) \not\approx \varphi_+\big(T(\boldsymbol{a})\big)$ by the Lagrangian Camel Theorem of [25]. Therefore, the connectedness requirement cannot be omitted in Theorem 5.1. The classification of product tori in $\mathcal{C}^{2n}(c)$ up to Hamiltonian isotopy may be difficult. Indeed, there might exist a symplectic embedding $\varphi\colon B^{2n}(b) \to \mathcal{C}^{2n}(c)$ whose image is so tangled up in the eye of $\mathcal{C}^{2n}(c)$ that $\varphi\big(T(\boldsymbol{a})\big)$ is Hamiltonian isotopic to neither of $\varphi_\pm\big(T(\boldsymbol{a})\big)$.

6. Proof of Theorem 1.5

6.1. Consider symplectic polar coordinates (ρ,θ) on $\dot{\mathbb{R}}^2 := \mathbb{R}^2 \setminus \{0\}$ defined by

$$(x,y) \;=\; \Big(\sqrt{\rho/\pi}\cos 2\pi\theta, \sqrt{\rho/\pi}\sin 2\pi\theta\Big), \quad \rho>0,\ \theta \in S^1 = \mathbb{R}/\mathbb{Z}.$$

For $s \in \mathbb{R}$ and $m \in \mathbb{Z}$, define the domain

$$\mathcal{D}_{m,s} = \{(\rho_1,\theta_1,\rho_2,\theta_2) \mid \rho_2 + s > m\rho_1\} \subset \mathbb{R}^4$$

and the map $\Psi_{m,s}\colon \mathcal{D}_{m,s} \to \mathbb{R}^4$,

$$\Psi_{m,s}(\rho_1,\theta_1,\rho_2,\theta_2) = (\rho_1, \theta_1 + m\theta_2, \rho_2 + s - m\rho_1, \theta_2).$$

The map $\Psi_{m,s}$ is a smooth symplectic embedding (for the same reasons as the map Ψ in the proof of Lemma 4.1).

Let (M,ω) be a symplectic manifold, and let $\varphi\colon B^{2n}(b_+) \to (M,\omega)$ be a symplectic chart. We denote by 0_{2j} the origin in $\mathbb{R}^{2j}$. The key step in the proof of Theorem 1.5 is the following proposition.

Proposition 6.1. *Let $k \geq 1$, $d_1,\dots,d_k,b_+ > 0$. Let $S \in \pi_2(M)$ be such that $s := \sigma(S)$ is positive and*

$$d_1 + \cdots + d_k + s < b_+.$$

Then there exist a neighbourhood U_k of the isotropic k-torus

$$T^k_{\mathtt{i}}(d_1,\dots,d_k) := 0_{2n-2k-2} \times T(d_1,\dots,d_{k-1}) \times 0_2 \times T(d_k)$$

in the open ball $\mathring{B}^{2n}(b_+)$ and a Hamiltonian symplectomorphism ψ_k of (M,ω) such that $(\psi_k \circ \varphi)(U_k) \subset \mathring{B}^{2n}_\varphi(b_+)$ and the map $\psi_k^\varphi := \varphi^{-1} \circ \psi_k \circ \varphi$ coincides with $\mathtt{id}_{2n-4} \times \Psi_{m,s}$ on U_k, where $m = c_1(S)$.

We will need the following lemma.

Lemma 6.2. *Given positive numbers $d_1,\dots,d_{k-1}$, for each $\varepsilon > 0$ there is a Hamiltonian flow $\{\Xi_t\}$, $t \in [0,1]$, on $\mathbb{R}^{2k}$ such that Ξ_1 maps the torus*

$$T = T(d_1,\dots,d_{k-1}) \times 0_2$$

into $\big(\mathring{B}^2(\varepsilon)\big)^k$ and Ξ_t maps T into $\mathring{B}^2(d_1+\varepsilon) \times \cdots \times \mathring{B}^2(d_{k-1}+\varepsilon) \times \mathring{B}^2(\varepsilon)$ for all t.

Proof. We start with the following

Lemma 6.3. *Given a positive number $d > 0$, for each $\varepsilon_0 > 0$ there exist $\delta = \delta(d, \varepsilon_0) > 0$ and a Hamiltonian flow $\{\Xi_t^{d,\varepsilon_0}\}$, $t \in [0,1]$, on $\mathbb{R}^4$ with the following properties:*
Ξ_t^{d,ε_0} maps $T(d) \times \mathring{B}^2(\delta)$ into $\mathring{B}^2(d+\varepsilon_0) \times \mathring{B}^2(\varepsilon_0)$ for all $t \in [0,1]$;
Ξ_1^{d,ε_0} maps $T(d) \times \mathring{B}^2(\delta)$ into $\mathring{B}^2(\varepsilon_0) \times \mathring{B}^2(\varepsilon_0)$.

Proof. For each $t \in [0,1]$ and for $\ell \in \mathbb{N}$, define the map $E_{t,\ell} \colon S^1 \to \mathbb{C}^2 = \mathbb{R}^4$ by

$$E_{t,\ell}(\theta) \,=\, \Big(\sqrt{(1-t)d/\pi}\; e^{2\pi i\theta},\; \sqrt{td/(\ell\pi)}\; e^{2\pi i \ell\theta} \Big).$$

Then $E_{0,\ell}$ is a diffeomorphism onto $T(d) \times 0_2$. For $t < 1$, the map $E_{t,\ell}$ is an embedding because its first component is. The integral over S^1 of the 1-form $E_{t,\ell}^*\lambda$, where $\lambda = x_1\,dy_1 + x_2\,dy_2$ is a primitive of ω_2, does not depend on t because

$$\int_{S^1} E_{t,\ell}^*\lambda \,=\, \int_{S^1} E_{t,\ell}^*(x_1\,dy_1) + \int_{S^1} E_{t,\ell}^*(x_2\,dy_2) \,=\, (1-t)d + td \,=\, d.$$

It follows that for each $q \in \,]0,1[$ there is a Hamiltonian flow $\{\Phi_t^{q,\ell}\}$, $t \in [0,1]$, such that $\Phi_t^{q,\ell}(T(d) \times 0_2) = E_{qt,\ell}(T(d) \times 0_2)$ for all $t \in [0,1]$. The absolute value of the first component of the map $E_{t,\ell}$ is decreasing with respect to t; the second component of $E_{t,\ell}$ tends uniformly to zero as $\ell \to \infty$. Therefore, after choosing ℓ large enough, we can assume that the tori $E_{t,\ell}(T(d) \times 0_2)$ are contained in $B^2(d) \times \mathring{B}^2(\varepsilon_0)$ for all $t \in [0,1]$ and that the torus $E_{1,\ell}(T(d) \times 0_2)$ is contained in $\mathring{B}^2(\varepsilon_0) \times \mathring{B}^2(\varepsilon_0)$. Then, after choosing q sufficiently close to 1, we can achieve that the torus $E_{q,\ell}(T(d) \times 0_2) = \Phi_1^{q,\ell}(T(d) \times 0_2)$ is contained in $\mathring{B}^2(\varepsilon_0) \times \mathring{B}^2(\varepsilon_0)$ as well. Let $\{\Xi_t^{d,\varepsilon_0} = \Phi_t^{q,\ell}\}$. By continuity, there exists $\delta = \delta(d, \varepsilon_0) > 0$ such that Ξ_t^{d,ε_0} maps $T(d) \times \mathring{B}^2(\delta)$ into $\mathring{B}^2(d+\varepsilon_0) \times \mathring{B}^2(\varepsilon_0)$ for all $t \in [0,1]$, and Ξ_1^{d,ε_0} maps $T(d) \times \mathring{B}^2(\delta)$ into $\mathring{B}^2(\varepsilon_0) \times \mathring{B}^2(\varepsilon_0)$. □

If $k = 2$, then Lemma 6.2 immediately follows from Lemma 6.3. Otherwise, applying Lemma 6.3 $k-1$ times, we construct positive numbers

$$\varepsilon_1 = \min\big(\delta(d_{k-1}, \varepsilon), \varepsilon\big),\; \varepsilon_2 = \min\big(\delta(d_{k-2}, \varepsilon_1), \varepsilon\big),\; \dots,\; \varepsilon_{k-1} = \min\big(\delta(d_1, \varepsilon_{k-2}), \varepsilon\big)$$

and Hamiltonian flows $\{\Xi_t^{d_{k-1},\varepsilon}\}, \{\Xi_t^{d_{k-2},\varepsilon_1}\}, \dots, \{\Xi_t^{d_1,\varepsilon_{k-2}}\}$ with the prescribed properties. Consider the Hamiltonian flows $\{\Phi_t^1\}, \{\Phi_t^2\}, \dots, \{\Phi_t^{k-1}\}$ on $\mathbb{R}^{2k}$ such that

$$\Phi_t^1 = \mathtt{id}_{2k-4} \times \Xi_t^{d_{k-1},\varepsilon},\; \Phi_t^2 = \mathtt{id}_{2k-6} \times \Xi_t^{d_{k-2},\varepsilon_1} \times \mathtt{id}_2, \dots, \Phi_t^{k-1} = \Xi_t^{d_1,\varepsilon_{k-2}} \times \mathtt{id}_{2k-4}\,.$$

For each $j \in \{1, \dots, k-1\}$, we have

$$\Phi_t^j\big(T(d_1, \dots, d_{k-j}) \times \mathring{B}^2(\varepsilon_j)\big) \times \big(\mathring{B}^2(\varepsilon)\big)^j \subset T(d_1, \dots, d_{k-j-1}) \times \mathring{B}^2(d_{k-j} + \varepsilon_{j-1}) \times \big(\mathring{B}^2(\varepsilon)\big)^j$$

for all $t \in [0,1]$, and

$$\Phi_1^j\big(T(d_1, \dots, d_{k-j}) \times \mathring{B}^2(\varepsilon_j)\big) \times \big(\mathring{B}^2(\varepsilon)\big)^j \subset T(d_1, \dots, d_{k-j-1}) \times \mathring{B}^2(\varepsilon_{j-1}) \times \big(\mathring{B}^2(\varepsilon)\big)^j,$$

where $\varepsilon_0 = \varepsilon$. Concatenating the flows $\{\Phi_t^1\}, \{\Phi_t^2\}, \dots, \{\Phi_t^{k-1}\}$ (and reparametrizing the result to make it smoothly depending on t), we obtain the required flow $\{\Xi_t\}$. □

6.2. *Proof of Proposition 6.1 for $k = 1$.* Denote $\mathcal{D} = \mathbb{R}^{2n-4} \times \mathcal{D}_{m,s}$,

$$\Psi \;=\; \mathtt{id}_{2n-4} \times \Psi_{m,s} \colon \mathcal{D} \to \mathbb{R}^{2n}.$$

Let $e_1 = d_1 + s$. Consider the maps $f_0, f_1 \colon S^1 \to \mathbb{R}^{2n}$,

$$f_0(\zeta) = 0_{2n-2} \times \big(\rho = d_1, \theta = \zeta\big), \quad f_1(\zeta) = 0_{2n-2} \times \big(\rho = e_1, \theta = \zeta\big).$$

We have $T^1_{\mathtt{i}}(d_1) = f_0(S^1)$, $T^1_{\mathtt{i}}(e_1) = f_1(S^1)$, and $\Psi \circ f_0 = f_1$. Let $f_0^\varphi = \varphi \circ f_0$, $f_1^\varphi = \varphi \circ f_1$.

First we prove that there is $\hat{\psi}_1 \in \mathrm{Ham}(M, \omega)$ such that $\hat{\psi}_1 \circ f_0^\varphi = f_1^\varphi$. Denote $Z = [0,1] \times S^1$. Consider the map $F \colon Z \to \mathbb{R}^{2n}$,

$$F(v, \zeta) = 0_{2n-2} \times \big(\rho = d_1 + vs, \theta = \zeta\big).$$

We have $f_0 = F(0, \cdot)$, $f_1 = F(1, \cdot)$, and

$$\int_Z (\varphi \circ F)^* \omega \;=\; \int_Z F^* \omega_n \;=\; \int_{S_1} f_1^*(\rho d\theta) - \int_{S_1} f_0^*(\rho d\theta) \;=\; s.$$

Taking the connected sum of $\varphi \circ F$ with a map $S^2 \to M$ representing the class $-S$, we obtain a smooth map $\widehat{F} \colon Z \to M$ such that $\widehat{F}$ coincides with $\varphi \circ F$ at the boundary of Z (that is, $f_0^\varphi = \widehat{F}(0, \cdot)$, $f_1^\varphi = \widehat{F}(1, \cdot)$) and

$$\int_Z \widehat{F}^* \omega = 0.$$

Then, according to [13, Appendix A], there exists a Hamiltonian flow $\{\hat{\psi}_t\}$ on (M, ω) such that the map

$$\widetilde{F} \colon Z \to M, \quad (v, \zeta) \mapsto \hat{\psi}_v(f_0^\varphi(\zeta))$$

is homotopic to $\widehat{F}$ relative to the boundary. In particular, this implies

$$\hat{\psi}_1 \circ f_0^\varphi = f_1^\varphi = \varphi \circ \Psi \circ f_0,$$

as required. It follows that $\varphi^{-1} \circ \hat{\psi}_1 \circ \varphi|_{T^1_{\mathtt{i}}(d_1)} = \Psi|_{T^1_{\mathtt{i}}(d_1)}$. Pick a neighbourhood $W \subset \mathring{B}^{2n}(b_+)$ of the circle $T^1_{\mathtt{i}}(d_1)$ such that the maps $\psi_W := \varphi^{-1} \circ \hat{\psi}_1 \circ \varphi|_W$ and $\Psi|_W$ are well defined. We shall prove that there is a Hamiltonian symplectomorphism Φ with support in W and a neighbourhood U_1 of the circle $T^1_{\mathtt{i}}(d_1)$ in W such that

$$\Phi|_{U_1} \;=\; \psi_W^{-1} \circ \Psi|_{U_1}. \tag{6}$$

Then the symplectomorphism $\psi_1 \in \mathrm{Ham}(M, \omega)$ that coincides with $\hat{\psi}_1 \circ \varphi \circ \Phi \circ \varphi^{-1}$ on $\varphi(W)$ and coincides with $\hat{\psi}_1$ outside $\varphi(W)$ will satisfy $\varphi^{-1} \circ \psi_1 \circ \varphi|_{U_1} = \Psi|_{U_1}$ as required.

Trivialize the tangent bundle of $\mathbb{R}^{2n-2} \times \dot{\mathbb{R}}^2$ using the symplectic frame

$$\xi = \big(\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_{n-1}}, \partial_{y_{n-1}}, \partial_{\rho_n}, \partial_{\theta_n}\big).$$

Denote by

$$\eta_w \colon \mathbb{R}^{2n} \to T_w\big(\mathbb{R}^{2n-2} \times \dot{\mathbb{R}}^2\big), \quad w \in \mathbb{R}^{2n-2} \times \dot{\mathbb{R}}^2$$

the corresponding trivialization maps. Let $\mathrm{Sp}(2n)$ denote the group of linear symplectomorphisms of $\mathbb{R}^{2n}$. Consider the loop

$$g\colon S^1 \to \mathrm{Sp}(2n), \quad g(\zeta) = \eta^{-1}_{f_0(\zeta)} \circ d\left(\psi_W^{-1} \circ \Psi\right) \circ \eta_{f_0(\zeta)}.$$

Recall that the fundamental group of $\mathrm{Sp}(2n)$ is isomorphic to $\mathbb{Z}$; this gives rise to a function μ called the *Maslov index* assigning to each continuous map $S^1 \to \mathrm{Sp}(2n)$ an integer (see [18, p. 48]).

Lemma 6.4. *The Maslov index of g vanishes.*

Proof. Define the maps $g_0, g_1\colon S^1 \to \mathrm{Sp}(2n)$,

$$g_0(\zeta) = \eta^{-1}_{f_1(\zeta)} \circ d\Psi \circ \eta_{f_0(\zeta)}, \quad g_1(\zeta) = \eta^{-1}_{f_1(\zeta)} \circ d\psi_W \circ \eta_{f_0(\zeta)}.$$

Since μ is additive with respect to the multiplication in $\mathrm{Sp}(2n)$ [18, Theorem 2.29], we have $\mu(g) = \mu(g_0) - \mu(g_1)$. By the definition of Ψ, we have $g_0(\zeta) = \mathtt{id}_{2n-4} \times A_\zeta \times \mathtt{id}_2$, where A_ζ acts on $\mathbb{C} = \mathbb{R}^2$ as complex multiplication by $e^{2\pi i m\zeta}$. Hence, according to [18, p. 49], $\mu(g_0) = m$.

In order to compute the Maslov index of g_1, consider the torus K constructed from two copies, Σ_1 and Σ_2, of the annulus $Z = [0,1] \times S^1$ by gluing together the respective boundary components. Define the map $u\colon K \to M$ that coincides with $\varphi \circ F$ on Σ_1, and with $\widehat{F}$ on Σ_2. Orient K by the volume form $dv \wedge d\zeta$ on Σ_2. Then the homology class of $u(K)$ is S. Consider the symplectic vector bundle u^*TM over K. Trivialize it over Σ_1 by means of the frame $\varphi_*\xi$, and over Σ_2, at the point (v,ζ), by means of the frame $(\hat{\psi}_v \circ \varphi)_*\xi$. Then it follows from [18, p. 75] that $\mu(g_1) = c_1(u(K)) = m$. Hence $\mu(g) = 0$. □

Denote by $\mathrm{Sp}_1(2n)$ the subgroup of the group $\mathrm{Sp}(2n)$ consisting of the maps sending the vector $(0,\dots,0,1)$ to itself. The loop g takes values in $\mathrm{Sp}_1(2n)$. By Lemma 6.4, g is contractible in $\mathrm{Sp}(2n)$. We claim that it is also contractible in $\mathrm{Sp}_1(2n)$. Indeed, the inclusion $i\colon \mathrm{Sp}_1(2n) \hookrightarrow \mathrm{Sp}(2n)$ is the fiber of the smooth fibration

$$\pi\colon \mathrm{Sp}(2n) \to \mathbb{R}^{2n} \setminus \{0\}, \quad A \mapsto A(0,\dots,0,1).$$

It follows from the long exact sequence of π that i induces an isomorphism of fundamental groups when $n \geq 2$. Thus there is a smooth family of maps $g^t\colon S^1 \to \mathrm{Sp}_1(2n)$, $t \in [0,1]$, such that $g^0 = \mathtt{id}$ and $g^1 = g$.

There is a linear isomorphism I from the space of quadratic forms on $\mathbb{R}^{2n}$ to the Lie algebra $\mathtt{sp}(2n)$ of the Lie group $\mathrm{Sp}(2n)$ that assigns to a quadratic form h the Hamiltonian vector field generated by h. The quadratic forms that vanish on the line $\{(0,\dots,0,\cdot)\}$ are isomorphically mapped by I to the Lie algebra $\mathtt{sp}_1(2n)$ of $\mathrm{Sp}_1(2n)$. From the family $\{g^t\}$ we construct a smooth family of Hamiltonian functions $\{H_t\}$ with support in W such that

$$\eta_w^{-1}\left(d^2(H_t)\right) = I^{-1}\left(\dot{g}^t(\theta_n)\right)$$

for all $w = (x_1, y_1, \dots, x_{n-1}, y_{n-1}, \rho_n, \theta_n) \in T^1_{\mathtt{i}}(d_1)$, $t \in [0,1]$. Then the time 1 flow Φ_+ generated by $\{H_t\}$ fixes each point $w \in T^1_{\mathtt{i}}(d_1)$ and has the same differential as $\psi_W^{-1} \circ \Psi$ at w.

The symplectomorphism $\Upsilon := \Phi_+^{-1} \circ \psi_W^{-1} \circ \Psi$ fixes $T^1_{\mathtt{i}}(d_1)$ pointwise and satisfies $d\Upsilon(w) = \mathtt{id}$ for all $w \in T^1_{\mathtt{i}}(d_1)$. We shall prove that there is a Hamiltonian symplectomorphism Φ_1 with support in W coinciding with Υ near $T^1_{\mathtt{i}}(d_1)$. Then $\Phi = \Phi_+ \circ \Phi_1$ is as required.

To construct Φ_1, we use generating functions (cf. [1, Section 48], [10, Appendix A.1]). Consider the graph $\Gamma \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ of the map Υ. Denote by $T^\times \subset \Gamma$ the circle consisting of the points (w, w), where $w \in T^1_{\mathtt{i}}(d_1)$. Denote by $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$ the symplectic coordinates on the first copy of $\mathbb{R}^{2n}$, and by $p' = (p'_1, \dots, p'_n), q = (q'_1, \dots, q'_n)$ those on the second copy. By construction, Γ is tangent to the diagonal $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ along $T^\times$. Hence there is a tubular neighbourhood V of $T^\times$ in Γ such that the map

$$\tau \colon V \to \mathbb{R}^{2n}, \quad (p, q, p', q') \mapsto (p', q)$$

is a diffeomorphism onto a neigbourhood U of $T^1_{\mathtt{i}}(d_1)$ in W. Since Υ is symplectic, V is Lagrangian with respect to the symplectic form

$$\Omega = -dp \wedge dq + dp' \wedge dq' = dq \wedge dp + dp' \wedge dq'.$$

The 1-forms $\alpha = -p dq + p' dq'$, $\alpha' = q dp + p' dq'$ satisfy $d\alpha = d\alpha' = \Omega$ and $\alpha = \alpha' - d(pq)$. Thus the restrictions of α and α' to V are closed. They are exact because the restriction of α to the diagonal Δ, and hence to the circle $T^\times \subset V \cap \Delta$, vanishes. Let $h \colon V \to \mathbb{R}$ be a primitive of α'. Define $F \colon \tau(V) \to \mathbb{R}$, $F = h \circ \tau^{-1}$. Then F is a generating function for V, namely, V is given by the equations

$$q = \frac{\partial F(p', q)}{\partial p'}, \quad p' = \frac{\partial F(p', q)}{\partial q}.$$

Note that $p'q$ is a generating function for Δ.

Since Γ is tangent to Δ along $T^\times$, the functions $F(p', q)$ and $p'q$ have the same respective first and second partial derivatives at the points of the circle $T^1_{\mathtt{i}}(d_1) = \tau(T^\times)$. Thus the function $f(p', q) := F(p', q) - p'q$ is C^2 small near $T^1_{\mathtt{i}}(d_1)$, and there exists a family of C^∞ smooth functions $f_\delta \colon \mathbb{R}^{2n} \to \mathbb{R}$, defined for sufficiently small positive δ, such that the function f_δ has support in the δ-neighbourhood W_δ of $T^1_{\mathtt{i}}(d_1)$, coincides with f on a smaller neighbourhood of $T^1_{\mathtt{i}}(d_1)$, and tends to zero in the C^2 topology as δ tends to zero. (To explicitly construct such a family, we can proceed as follows. Pick a family of smooth compactly supported functions $\lambda_\delta \colon\ [0, \delta[\to [0, \delta[$ such that λ_δ is identity near 0 and its first and second derivatives are bounded uniformly over δ. Given $x \in W_\delta$, denote by x_0 the point of $T^1_{\mathtt{i}}(d_1)$ closest to x and draw the ray starting at x_0 and passing through x. Let $G_\delta \colon W_\delta \to W_\delta$ be the map that sends x to the point y such that y lies on this ray and $\mathrm{dist}(y, x_0) = \lambda_\delta(\mathrm{dist}(x, x_0))$. Define f_δ to coincide with $f \circ G_\delta$ on W_δ.)

Denote by L^t_δ the Lagrangian submanifold in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ defined by the generating function $p'q + t f_\delta(p', q)$. Picking δ sufficiently small, we can assume that each of the manifolds L^t_δ is sufficiently C^1 close to Δ and hence is a graph of a compactly supported symplectomorphism Φ_t. The symplectomorphism Φ_1 is Hamiltonian because $\Phi_0 = \mathtt{id}$ and $H^1(\mathbb{R}^{2n}) = 0$. Making δ smaller if necessary, we can assume that each Φ_t has support in W. Since $p'q + f_\delta(p', q)$ coincides with F near $T^1_{\mathtt{i}}(d_1)$, the symplectomorphisms Φ_1 and Υ also coincide near $T^1_{\mathtt{i}}(d_1)$. Thus Φ_1 is as required, which concludes the proof of Proposition 6.1 for

$k = 1$. □

6.3. *Proof of Proposition 6.1 for $k > 1$.* Applying Proposition 6.1 for $k = 1$ to the circle $T^1_{\mathtt{i}}(d_k)$, we obtain a neighbourhood U_1 of $T^1_{\mathtt{i}}(d_k)$ and a Hamiltonian symplectomorphism ψ_1 such that $\psi_1^\varphi|_{U_1} = \Psi|_{U_1}$. We shall construct a neighbourhood $U_k \subset \mathcal{D}$ of the torus $T^k_{\mathtt{i}} := T^k_{\mathtt{i}}(d_1, \dots, d_k)$ and Hamiltonian symplectomorphisms $\Theta, \Theta_\star$ with support in $\mathring{B}^{2n}(b_+)$ such that

$$\Theta(U_k) \subset U_1, \qquad \Psi \circ \Theta|_{U_k} = \Theta_\star \circ \Psi|_{U_k}.$$

Denote by Θ^φ (resp. $\Theta^\varphi_\star$) the Hamiltonian symplectomorphism of (M, ω) that coincides with $\varphi \circ \Theta \circ \varphi^{-1}$ (resp. $\varphi \circ \Theta_\star \circ \varphi^{-1}$) on $B^{2n}_\varphi(b_+)$ and with the identity elsewhere. The symplectomorphism $\psi_k = (\Theta^\varphi_\star)^{-1} \circ \psi_1 \circ \Theta^\varphi$ will then have the required property since

$$\varphi^{-1} \circ \psi_k \circ \varphi|_{U_k} \;=\; \Theta_\star^{-1} \circ \psi_1^\varphi \circ \Theta|_{U_k} \;=\; \Theta_\star^{-1} \circ \Psi \circ \Theta|_{U_k} \;=\; \Psi|_{U_k}.$$

It remains to construct Θ and $\Theta_\star$. Let $\varepsilon > 0$. Applying Lemma 6.2, we obtain a Hamiltonian flow $\{\Xi_t\}$ on $\mathbb{R}^{2n}$ such that Ξ_1 maps the torus $T = T(d_1, \dots, d_{k-1}) \times 0_2$ into $\left(\mathring{B}^2(\varepsilon)\right)^k$ and

$$\Xi_t(T) \subset \mathring{B}^2(d_1 + \varepsilon) \times \cdots \times \mathring{B}^2(d_{k-1} + \varepsilon) \times \mathring{B}^2(\varepsilon) \;\text{ for all }\; t \in [0,1]. \tag{7}$$

Consider the Hamiltonian flow

$$\left\{\mathrm{P}_t = \mathtt{id}_{2n-2k-2} \times \Xi_t \times \mathtt{id}_2\right\}, \; t \in [0,1],$$

on $\mathbb{R}^{2n}$. Let $b' = b_+ - s$. Clearly, the torus $T^k_{\mathtt{i}}$ is contained in $\mathcal{D} \cap \mathring{B}^{2n}(b')$. We claim that by choosing ε sufficiently small we can achieve that P_t maps $T^k_{\mathtt{i}}$ into $\mathcal{D} \cap \mathring{B}^{2n}(b')$ for all $t \in [0,1]$, and that P_1 maps $T^k_{\mathtt{i}}$ into U_1. Indeed, if $m\varepsilon < d_k$, then the set

$$\mathring{B}^2(\varepsilon) \times T(d_k) = \{\rho_1 < \varepsilon, \, \rho_2 = d_k\}$$

is contained in $\mathcal{D}_{m,s}$. It follows from (7) that for all t the torus $\mathrm{P}_t(T^k_{\mathtt{i}})$ is contained in $\mathbb{R}^{2n-4} \times \mathring{B}^2(\varepsilon) \times T(d_k)$, and hence in $\mathcal{D}$. If $d_1 + \cdots + d_k + k\varepsilon < b'$, then it follows from (7) that $\mathrm{P}_t(T^k_{\mathtt{i}}) \subset \mathring{B}^{2n}(b')$ for all t. Finally, for ε such that $0_{2n-2k-2} \times \left(\mathring{B}^2(\varepsilon)\right)^k \times T(d_k)$ is a subset of U_1, we have $\mathrm{P}_1(T^k_{\mathtt{i}}) \subset U_1$.

It follows from the definition of the map Ψ that $\Psi(\mathrm{P}_t(T^k_{\mathtt{i}}))$ is contained in $\mathring{B}^{2n}(b_+)$ for all $t \in [0,1]$. Therefore, there is an open set $W \subset \mathcal{D} \cap \mathring{B}^{2n}(b')$ that contains all the tori $\Psi(\mathrm{P}_t(T^k_{\mathtt{i}}))$ and satisfies $\Psi(W) \subset \mathring{B}^{2n}(b_+)$. Then there exists a neighbourhood U_k of the torus $T^k_{\mathtt{i}}$ such that $\mathrm{P}_t(U_k) \subset W$ for all t, and $\mathrm{P}_1(U_k) \subset U_1$.

Applying to $\{\mathrm{P}_t\}$ an appropriate cut-off, we construct a Hamiltonian flow $\{\mathrm{P}'_t\}$, $t \in [0,1]$, with support in W such that $\mathrm{P}'_t|_{U_k} = \mathrm{P}_t|_{U_k}$ for all t and $\mathrm{P}'_1(U_k) \subset U_1$. Define the Hamiltonian flow $\{\mathrm{P}^\star_t\}$, $t \in [0,1]$, with support in $\Psi(W) \subset \mathring{B}^{2n}(b_+)$ by $\mathrm{P}^\star_t = \Psi \circ \mathrm{P}'_1 \circ \Psi^{-1}$. Then $\Theta = \mathrm{P}'_1$ and $\Theta_\star = \mathrm{P}^\star_1$ are as required. □

6.4. *Proof of Theorem 1.5.* It suffices to prove the theorem under the additional assumption that $d_j = e_j$ for $j < k$. Indeed, in view of Theorem 1.1 (i), the claim will then also hold for vectors that differ at only one component; after that the general case follows by changing one component at a time.

We extend the symplectic chart φ from $B^{2n}(b)$ to a larger ball $B^{2n}(b_+)$ with $b_+ > b$, and keep the letter φ for this extension. For $\boldsymbol{d}' = (d'_1, \dots, d'_k)$, we abbreviate $T_\varphi(a, \dots, a, a + d'_1, \dots, a + d'_k)$ to $T_{\varphi,a}(\boldsymbol{d}')$. Given $\tau \in \big[0, \min(c, b_+ - b)\big[$, denote by $\mathcal{V}_\tau$ the subset of $\mathbb{R}^k$ formed by vectors $(d_1, \dots, d_k)$ such that $d_1 + \dots + d_k \leq b + \tau$ and $d_j \geq c - \tau$ for all $j \in \{1, \dots, k\}$. Pick $\delta \in \big]0, \min(c, b_+ - b)\big[$. Recall that $\sigma_a(S) = \sigma(S) - c_1(S)a$.

Lemma 6.5. *Let $S \in \pi_2(M)$. There exists $A_S > 0$ such that for each $a \in \,]0, A_S]$ and for each pair of vectors*

$$\boldsymbol{d} = (d_1, \dots, d_{k-1}, d_k)\,, \quad \boldsymbol{d}_S = (d_1, \dots, d_{k-1}, d_k + \sigma_a(S))$$

belonging to $\mathcal{V}_\delta$, we have $T_{\varphi,a}(\boldsymbol{d}) \approx T_{\varphi,a}(\boldsymbol{d}_S)$.

Proof. Denote $s = \sigma(S)$, $m = c_1(S)$. Assume first that $s \geq 0$. It follows from Proposition 6.1 and the definition of the map $\Psi_{m,s}$ that for each $\boldsymbol{d} \in \mathcal{V}_\delta$ there exist a neighbourhood U of the isotropic k-torus $T^k_{\mathtt{i}}(\boldsymbol{d})$ in $\mathring{B}^{2n}(b_+)$ and a map $\psi \in \mathrm{Ham}(M, \omega)$ such that for every torus $T(a_1, \dots, a_n)$ contained in U we have

$$\psi\big(T_\varphi(a_1, \dots, a_{n-1}, a_n)\big) = T_\varphi(a_1, \dots, a_{n-1}, a_n + s - m a_{n-1}).$$

Therefore, by Theorem 1.1 (i), for each $\boldsymbol{d} \in \mathcal{V}_\delta$ there are a positive number $A_{S,\boldsymbol{d}}$ and a neighbourhood $W_{S,\boldsymbol{d}}$ of $\boldsymbol{d}$ in $\mathcal{V}_\delta$ such that for each $\boldsymbol{d}' \in W_{S,\boldsymbol{d}}$ and each $a \in \,]0, A_{S,\boldsymbol{d}}]$ we have $T_{\varphi,a}(\boldsymbol{d}') \approx T_{\varphi,a}(\boldsymbol{d}'_S)$.

Since $\mathcal{V}_\delta$ is compact, there are $\boldsymbol{d}^{(1)}, \dots, \boldsymbol{d}^{(l)} \in \mathcal{V}_\delta$ such that the sets $W_{S,\boldsymbol{d}^{(j)}}$ cover $\mathcal{V}_\delta$. Let A_S be the smallest of the numbers $A_{S,\boldsymbol{d}^{(j)}}$. Then $T_{\varphi,a}(\boldsymbol{d}) \approx T_{\varphi,a}(\boldsymbol{d}_S)$ for each $\boldsymbol{d} \in \mathcal{V}_\delta$ and each $a \in \,]0, A_S]$. In particular, $T_{\varphi,a}(\boldsymbol{d}) \approx T_{\varphi,a}(\boldsymbol{d}_S)$ for each $a \in \,]0, A_S]$ when $\boldsymbol{d}, \boldsymbol{d}_S \in \mathcal{V}_\delta$. The latter statement is invariant under changing the sign of S, and therefore we can drop the assumption that $s \geq 0$. □

Assume first that (M, ω) is not special. Let $S_1, \dots, S_r$ be elements of $\pi_2(M)$ such that their classes form the basis of the free Abelian group $\pi_2(M)/\big(\ker\sigma \cap \ker c_1\big)$. We can assume that $r \geq 1$, otherwise there is nothing to prove. Consider the free Abelian group $\sigma\,(\pi_2(M))$. If it is trivial, then $r = 1$. If its rank is 1, then $r = 1$ (otherwise (M, ω) would be special). If the rank of this group is greater than 1, then $r \geq 2$ and we can choose $S_1, \dots, S_r$ such that for all $j \in \{1, \dots, r\}$ the numbers $s_j = \sigma(S_j)$ satisfy the inequality $|s_j| < \delta$. Denote $m_j = c_1(S_j)$. Pick $A > 0$ such that for all $j \in \{1, \dots, r\}$ we have

$$A \leq A_{S_j}, \quad |s_j - m_j A| < \delta.$$

If (M, ω) is special, we set $r = 1$, $S_1 = S_0$ (or $S_1 = -S_0$), and $A = A_{S_1}$.

Let $a \in \,]0, A]$. Let

$$\boldsymbol{d} = (d_1, \dots, d_{k-1}, d_k)\,, \quad \boldsymbol{e} = (d_1, \dots, d_{k-1}, e_k)$$

be vectors in $\mathcal{V}_\delta$. We assume that the difference $d_k - e_k$ is an element of $G_a = \sigma_a(\pi_2(M))$ if (M, ω) is not special, and an element of $G_a(S_0) = \sigma_a(\langle S_0 \rangle))$ if (M, ω) is special. Hence there are $n_1, \dots, n_r \in \mathbb{Z}$ such that

$$e_k - d_k \;=\; \sum_{j=1}^{r} n_j \sigma_a(S_j) \;=\; \sum_{j=1}^{r} n_j (s_j - m_j a).$$

After changing the signs of S_j if necessary, we can assume that all coefficients n_j are nonnegative. We need to prove that $T_{\varphi,a}(\boldsymbol{d}) \approx T_{\varphi,a}(\boldsymbol{e})$.

Let $u_1, \dots, u_N$ be a sequence of numbers such that for each $j \in \{1, \dots, r\}$ exactly n_j of them equal $s_j - m_j a$. It gives rise to the sequence $q_0, q_1, \dots, q_N$, where $q_0 = d_k$, $q_l = d_k + \sum_{i=1}^{l} u_i$ for all $l \in \{1, \dots, N\}$ (and hence $q_N = e_k$). Without loss of generality, we can assume that $d_k < e_k$. If

$$q_l \in [d_k - \delta, e_k + \delta] \;\text{ for all }\; l \in \{1, \dots, N\}, \tag{8}$$

then each of the vectors $\boldsymbol{q}_l = (d_1, \dots, d_{k-1}, q_l)$ belongs to $\mathcal{V}_\delta$. Since $a \le A_{S_j}$ for all j, it then follows from Lemma 6.5 that

$$T_{\varphi,a}(\boldsymbol{d}) = T_{\varphi,a}(\boldsymbol{q}_0) \approx T_{\varphi,a}(\boldsymbol{q}_1) \approx \cdots \approx T_{\varphi,a}(\boldsymbol{q}_{N-1}) \approx T_{\varphi,a}(\boldsymbol{q}_N) = T_{\varphi,a}(\boldsymbol{e}).$$

It remains to show that the sequence $u_1, \dots, u_N$ can be chosen to satisfy (8). For $r = 1$, there is no choice involved in the construction of the sequence, and all q_l belong to $[d_k, e_k]$. Let $r > 1$. Then $|s_j - m_j a| < \delta$ for all j since $|s_j| < \delta$ and $|s_j - m_j A| < \delta$. We choose the numbers u_l in succession, using the following rule: if $q_{l-1} > e_k$, then $u_l < 0$, and if $q_{l-1} < d_k$, then $u_l > 0$. Then (8) will hold true. This completes the proof of Theorem 1.5 □

Appendix A. Areas of holomorphic curves in a hyperannulus

For $r > 0$, denote by B_r (resp. $\mathring{B}_r$) the closed (resp. open) ball of radius r in the complex vector space $\mathbb{C}^n$ centred at the origin. Denote $B_0 = \{0\}$. In this appendix, we prove the following

Theorem A.1. *Let $r_+ > r_- \ge 0$. Let V be a holomorphic curve (a 1-dimensional analytic subvariety) in the hyperannulus $\mathring{B}_{r_+} \setminus B_{r_-}$ such that the closure of V intersects ∂B_{r_-}. Then the area of V is at least $\pi(r_+^2 - r_-^2)$.*

If the area equals $\pi(r_+^2 - r_-^2)$, then V is the intersection of a complex line in $\mathbb{C}^n$ with the hyperannulus.

In the particular case where $r_- = 0$, Theorem A.1 is equivalent to the 1-dimensional version of the Lelong theorem that gives a lower bound for the areas of holomorphic curves in a ball passing through the centre [14, 26].

Let $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$ be the standard coordinates on $\mathbb{C}^n$. Consider the 1-form $\alpha_n = \sum_{j=1}^{n} (x_j\, dy_j - y_j\, dx_j)$ on $\mathbb{C}^n$.

Lemma A.2. *Let $\gamma\colon S^1 \to \mathbb{C}^n$ be a C^1-smooth curve. Then its length $\ell(\gamma)$ satisfies the inequality*

$$\ell^2(\gamma) \geq 2\pi \int_{S^1} \gamma^* \alpha_n.$$

Proof. For $n = 1$, this is the classical isoperimetric inequality, see [21]. The general case reduces to the 1-dimensional case as follows. Write $\gamma = (\rho_1, \ldots, \rho_n)$, where $\rho_1, \ldots, \rho_n$ are maps from $S^1 = \mathbb{R}/\mathbb{Z}$ to $\mathbb{C}$. By the Minkowski inequality (see [8], p. 146), we have

$$\begin{aligned} \ell^2(\gamma) &= \left(\int_{S^1} |\dot\gamma(t)|\,dt\right)^2 = \left(\int_{S^1} \sqrt{\textstyle\sum_{j=1}^n |\dot\rho_j(t)|^2}\,dt\right)^2 \\ &\geq \sum_{j=1}^n \left(\int_{S^1} |\dot\rho_j(t)|\,dt\right)^2 = \sum_{j=1}^n \ell^2(\rho_j), \end{aligned} \tag{A1}$$

where $|\cdot|$ denotes the length of the vector, and $\ell(\rho_j)$ is the length of the curve ρ_j. Since the isoperimetric inequality holds for the curves ρ_j, we have

$$\sum_{j=1}^n \ell^2(\rho_j) \geq 2\pi \sum_{j=1}^n \int_{S^1} \rho_j^* \alpha_1 = 2\pi \int_{S^1} \gamma^* \alpha_n.$$

□

Proof of Theorem A.1. Let Q be the subset of $]r_-, r_+[$ formed by those r for which V is transverse to ∂B_r. For reasons of analyticity, the complement of Q has no accumulation point in $]r_-, r_+[$. In particular, Q is of full measure in $]r_-, r_+[$. Consider the function $F\colon [r_-, r_+] \to \mathbb{R}$, where $F(r)$ is the area of the curve $V \cap (\mathring{B}_r \setminus B_{r_-})$. Then $F(r_-) = 0$ and F is monotone non-decreasing. We shall prove that for each $r \in Q$, the derivative $F'(r)$ exists and is not less than $2\pi r$. Since F is monotone non-decreasing, its derivative F' is measurable and $\int_{r_-}^{r_+} F'\,dr \leq F(r_+) - F(r_-)$, see Theorem 7.21 in [30]. (Actually, it is easy to check that F is continuous and hence the inequality is an equality.) Therefore,

$$F(r_+) = F(r_+) - F(r_-) \geq \int_{r_-}^{r_+} F'\,dr \geq \int_{r_-}^{r_+} 2\pi r\,dr = \pi(r_+^2 - r_-^2).$$

It follows from the maximum principle that V intersects each sphere $S_r = \partial B_r$. Let $r \in Q$. The set $V \cap S_r$ is the union of immersed circles $W_1, \ldots, W_m$, where $m \geq 1$. Parametrize these circles by immersions $\gamma_1, \ldots, \gamma_m$ of $S^1 = \mathbb{R}/\mathbb{Z}$ into S_r (such that, for each k, the image of γ_k is W_k, and γ_k is an embedding outside finitely many points). Consider the angle $\psi_k(t) \in [0, \pi/2]$ between the vector $i\dot\gamma_k(t)$ and the tangent space $T_{\gamma_k(t)} S_r$. Since $i\dot\gamma_k(t)$ is tangent to V and orthogonal to $\dot\gamma_k(t)$, the angle between the tangent spaces to V and S_r at the point $\gamma_k(t)$ also equals $\psi_k(t)$. Since V intersects S_r transversely, this angle is always positive. Denote by $s_k(t, \varepsilon)$ the oriented (having the same sign as ε) distance between the point $\gamma_k(t)$ and the sphere $S_{r+\varepsilon}$, measured along the direction of $i\dot\gamma_k(t)$. We

have $s_k(t,\varepsilon)=\varepsilon/\sin(\psi_k(t))+O(\varepsilon^2)$, and thus

$$F'(r)=\sum_{k=1}^{m}\int_{S^1}|\dot\gamma_k(t)|\frac{\partial s_k(t,\varepsilon)}{\partial\varepsilon}\,dt=\sum_{k=1}^{m}\int_{S^1}\frac{|\dot\gamma_k(t)|}{\sin(\psi_k(t))}\,dt.$$

We shall prove that

$$\text{(A2)}\qquad \int_{S^1}\frac{|\dot\gamma_k(t)|}{\sin(\psi_k(t))}\,dt\geq 2\pi r$$

for each k, and hence $F'(r)\geq 2\pi rm\geq 2\pi r$.

Consider the Euler vector field $\zeta=\sum_{j=1}^{n}(x_j\partial_{x_j}+y_j\partial_{y_j})$ and the vector field $\xi=\sum_{j=1}^{n}(x_j\partial_{y_j}-y_j\partial_{x_j})$ on $\mathbb{C}^n$. Let $q\in S_r$. Multiplication by i in the tangent space $T_q\mathbb{C}^n$ takes the hyperplane T_qS_r to the kernel $\ker_q\alpha_n$ of the 1-form $\alpha_n=\sum_{j=1}^{n}(x_jdy_j-y_jdx_j)$, and the vector ζ_q to the vector $\xi_q\in T_qS_r$. Since ζ_q is orthogonal to T_qS_r, the vector ξ_q is orthogonal to $\ker_q\alpha_n$. Fix $k\in\{1,\dots,m\}$. We write γ for γ_k and ψ for ψ_k. Because multiplication by i is an isometry, the angle between $\dot\gamma(t)$ and $\ker_{\gamma(t)}\alpha_n$ equals $\psi(t)$. Let $u(t)$ be the non-zero vector in $T_{\gamma(t)}S_r$ obtained by projecting $\dot\gamma(t)$ along the hyperplane $\ker_{\gamma(t)}\alpha_n$ onto the line containing $\xi_{\gamma(t)}$. We can assume that $u(t)$ is a positive multiple of $\xi_{\gamma(t)}$ for all t (after reversing the orientation of S^1 if necessary). Since $\alpha_n(\xi_q)=r^2=r|\xi_q|$ for each $q\in S_r$, we have $\alpha_n(u(t))=r|u(t)|$. Thus

$$\text{(A3)}\qquad \int_{S^1}\gamma^*\alpha_n=\int_{S^1}\alpha_n(u(t))\,dt=r\int_{S^1}|u(t)|\,dt.$$

We have $|u(t)|=\sin(\psi(t))\,|\dot\gamma(t)|$, and hence

$$\text{(A4)}\qquad \int_{S^1}\frac{|\dot\gamma(t)|}{\sin(\psi(t))}\,dt=\int_{S^1}\frac{|\dot\gamma(t)|^2}{|u(t)|}\,dt.$$

By the Cauchy–Schwarz inequality,

$$\text{(A5)}\qquad \int_{S^1}\frac{|\dot\gamma(t)|^2}{|u(t)|}\,dt\int_{S^1}|u(t)|\,dt\geq\left(\int_{S^1}|\dot\gamma(t)|\,dt\right)^2=\ell^2(\gamma),$$

where $\ell(\gamma)$ is the length of γ. Applying Lemma A.2 to γ and using (A3), we obtain the inequality

$$\text{(A6)}\qquad \ell^2(\gamma)\geq 2\pi\int_{S^1}\gamma^*\alpha_n=2\pi r\int_{S^1}|u(t)|\,dt.$$

Combining (A4), (A5), and (A6), we get (A2). This proves the first statement.

In order to prove the second statement, suppose that V has area $\pi(r_+^2-r_-^2)$. Then all inequalities in the above argument, including those in the proof of Lemma A.2, turn into equalities. We may assume that V is irreducible, by considering only one of the irreducible components of V (note that each component has positive area).

Pick $r\in Q$, and let $\gamma=(\rho_1,\dots,\rho_n)$ be an immersion parametrizing a component of $V\cap S_r$. We may assume, after applying to V a unitary transformation, that the complex line tangent to V at the point $\gamma(0)$ is parallel to the first coordinate axis. Then $\dot\rho_2(0)=\dots=\dot\rho_n(0)=0$. The Minkowski inequality (A1) is an equality, which implies that

the functions $\dot\rho_1, \dots, \dot\rho_n$ are proportional. Since there is a point where all of them except the first one vanish, the functions $\dot\rho_2, \dots, \dot\rho_n$ vanish everywhere. Therefore, the image of γ is contained in a complex line E parallel to the first coordinate axis. Consider the intersection E' of E with the hyperannulus. Because E' contains the image of γ, for reasons of analyticity E' is contained in the curve V. But V is irreducible, hence $V = E'$. This completes the proof. □

Appendix B. Existence of low admissible paths

Let $k \geq 2$. Given an ordered pair of different numbers $i, j \in \{1, \dots, k\}$, consider the operator P_{ij} (resp. M_{ij}, resp. I_{ij}) in $\mathrm{GL}(k; \mathbb{Z})$ that adds to the i-th component of a vector in $\mathbb{R}^k$ its j-th component (resp. subtracts from the i-th component the j-th component, resp. swaps the i-th component and the j-th component), and does not change the other components.

Denote by $\mathbb{R}_+$ the set of positive real numbers. A sequence $\boldsymbol{d} = \boldsymbol{d}^0, \boldsymbol{d}^1, \boldsymbol{d}^2, \dots, \boldsymbol{d}^\ell = \boldsymbol{e}$ of vectors in $\mathbb{R}^k_+$ is called an *admissible path* from $\boldsymbol{d}$ to $\boldsymbol{e}$ if for each s the vector $\boldsymbol{d}^{s+1}$ is obtained from $\boldsymbol{d}^s$ by the action of one of the operators P_{ij}, M_{ij}, I_{ij}. Given vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^k_+$, we write $\boldsymbol{v} \leq \boldsymbol{w}$ if there is a permutation σ of $\{1, \dots, k\}$ such that $v_i \leq w_{\sigma(i)}$ for all $i \in \{1, \dots, k\}$. This defines a partial order on $\mathbb{R}^k_+$. We say that a path $\boldsymbol{d} = \boldsymbol{d}^0, \boldsymbol{d}^1, \dots, \boldsymbol{d}^\ell = \boldsymbol{d}'$ is *low* if for each $s \in \{0, 1, \dots, l\}$ we have $\boldsymbol{d}^s \leq \boldsymbol{d}$ or $\boldsymbol{d}^s \leq \boldsymbol{d}'$. Given $\boldsymbol{u} = (u_1, \dots, u_k) \in \mathbb{R}^k$, we write

$$\langle \boldsymbol{u} \rangle = \langle u_1, \dots, u_k \rangle$$

for the free Abelian subgroup in $\mathbb{R}$ generated over $\mathbb{Z}$ by the numbers $u_1, \dots, u_k$.

The following theorem may be well known to specialists in number theory or geometric group theory, but we were unable to find it in the literature.

Theorem B.1. *Given $\boldsymbol{d} = (d_1, \dots, d_k)$ and $\boldsymbol{e} = (e_1, \dots, e_k)$ in $\mathbb{R}^k_+$ such that $\langle \boldsymbol{d} \rangle = \langle \boldsymbol{e} \rangle$, there is a low admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$.*

This appendix is devoted to the proof of Theorem B.1. We start with two remarks concerning low admissible paths. First, if the path $\boldsymbol{d}^0, \boldsymbol{d}^1, \dots, \boldsymbol{d}^{\ell-1}, \boldsymbol{d}^\ell$ is admissible, then the path $\boldsymbol{d}^\ell, \boldsymbol{d}^{\ell-1}, \dots, \boldsymbol{d}^1, \boldsymbol{d}^0$ is also admissible, because $M_{ij}^{-1} = P_{ij}$. Second, the concatenation of a low path from $\boldsymbol{d}$ to $\boldsymbol{d}'$ and a low path from $\boldsymbol{d}'$ to $\boldsymbol{d}''$ does not have to be low. However, the concatenation is always low when $\boldsymbol{d}' \leq \boldsymbol{d}$ or $\boldsymbol{d}' \leq \boldsymbol{d}''$.

Lemma B.2. *If $\boldsymbol{u}, \boldsymbol{u}' \in \mathbb{R}^l$ and $\langle \boldsymbol{u} \rangle = \langle \boldsymbol{u}' \rangle$, then there is $A \in \mathrm{GL}(l, \mathbb{Z})$ such that $A\boldsymbol{u} = \boldsymbol{u}'$.*

Proof. Consider the homomorphisms $h, h' \colon \mathbb{Z}^l \to \langle \boldsymbol{u} \rangle$, $h(\boldsymbol{n}) = (\boldsymbol{u}; \boldsymbol{n})$, $h'(\boldsymbol{n}) = (\boldsymbol{u}'; \boldsymbol{n})$, where $(\cdot;\cdot)$ is the scalar product on $\mathbb{R}^l$. Since h and h' are surjective homomorphisms of free Abelian groups, there are splittings $\mathbb{Z}^l = \ker(h) \oplus \Lambda = \ker(h') \oplus \Lambda'$, where Λ, Λ' are subgroups of $\mathbb{Z}^l$, and the restrictions of h, h' to Λ, Λ' respectively are isomorphisms onto $\langle \boldsymbol{u} \rangle$. Consider a homomorphism $B \colon \mathbb{Z}^l \to \mathbb{Z}^l$ such that $B|_{\Lambda'} = (h|_{\Lambda})^{-1} \circ h'|_{\Lambda'}$ and $B|_{\ker(h')}$ is an isomorphism from $\ker(h')$ onto $\ker(h)$. We have $B \in \mathrm{GL}(l, \mathbb{Z})$ and $h' = h \circ B$.

Define $A \in \mathrm{GL}(l, \mathbb{Z})$ to be the transpose of B. Then

$$(\boldsymbol{u}'; \boldsymbol{n}) \,=\, h'(\boldsymbol{n}) \,=\, h(B\boldsymbol{n}) \,=\, (\boldsymbol{u}; B\boldsymbol{n}) = (A\boldsymbol{u}; \boldsymbol{n})$$

for each $\boldsymbol{n} \in \mathbb{Z}^l$, and hence $A\boldsymbol{u} = \boldsymbol{u}'$. $\square$

Lemma B.3. *Given $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{R}^k_+$ such that $\langle \boldsymbol{d} \rangle = \langle \boldsymbol{e} \rangle$ and $\langle \boldsymbol{d} \rangle$ has rank 1, there is a low admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$.*

Proof. Let $d_0 > 0$ be such that $\langle d_0 \rangle = \langle \boldsymbol{d} \rangle$, and let $\boldsymbol{d}' = (d_0, \dots, d_0) \in \mathbb{R}^k_+$. By repeatedly applying to $\boldsymbol{d}$ and $\boldsymbol{e}$ operations M_{ij}, we construct admissible paths $\boldsymbol{d}, \boldsymbol{d}^1, \dots, \boldsymbol{d}^\ell = \boldsymbol{d}'$ and $\boldsymbol{e}, \boldsymbol{e}^1, \dots, \boldsymbol{e}^m = \boldsymbol{d}'$, where $\boldsymbol{d} \geq \boldsymbol{d}^1 \geq \cdots \geq \boldsymbol{d}^\ell$ and $\boldsymbol{e} \geq \boldsymbol{e}^1 \geq \cdots \geq \boldsymbol{e}^m$. Then $\boldsymbol{d}, \dots, \boldsymbol{d}', \dots, \boldsymbol{e}$ is a low admissible path. $\square$

Lemma B.4. *Given $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{R}^2_+$ such that $\langle \boldsymbol{d} \rangle = \langle \boldsymbol{e} \rangle$ and $\langle \boldsymbol{d} \rangle$ has rank 2, there is a low admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$.*

Proof. By Lemma B.2, there exists $A \in \mathrm{GL}(2; \mathbb{Z})$ such that $A(\boldsymbol{d}) = \boldsymbol{e}$. The following lemma then shows that there is an admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$.

Lemma B.5. *Given $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{R}^2_+$ and $A \in \mathrm{GL}(2; \mathbb{Z})$ such that $A(\boldsymbol{d}) = \boldsymbol{e}$, there is an admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$.*

Proof. Let $\mathcal{E}$ be the set of matrices in $\mathrm{GL}(2; \mathbb{Z})$ with one entry 0 and three entries in $\{1, -1\}$. The set $\mathcal{E}$ generates the group $\mathrm{GL}(2; \mathbb{Z})$. Indeed, $\mathrm{GL}(2; \mathbb{Z})$ is generated by the three matrices

$$P \,=\, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad I \,=\, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Q_1 \,=\, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(see e.g. [5, p. 43]), and we have $P \in \mathcal{E}$,

$$I = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} P, \qquad Q_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} P.$$

Moreover, $\mathcal{E}$ is closed under taking inverses. Hence we can write $A = E_\ell \cdots E_2 E_1$, where $E_s \in \mathcal{E}$ for all $s \in \{1, \dots, \ell\}$. Let $\boldsymbol{d}^0 = \boldsymbol{d}$, and let

$$\widetilde{\boldsymbol{d}}^s = (x^s, y^s) = E_s \cdots E_2 E_1(\boldsymbol{d}), \quad \boldsymbol{d}^s = (|x^s|, |y^s|)$$

for each $s > 0$. The components of $\boldsymbol{d}^s$ cannot vanish because they generate the free Abelian group $\langle \boldsymbol{d} \rangle$ of rank 2. Hence $\boldsymbol{d}^s \in \mathbb{R}^2_+$ for each s. It suffices to construct for each $s \in \{1, \dots, \ell\}$ an admissible path from $\boldsymbol{d}^{s-1}$ to $\boldsymbol{d}^s$. Denote by $\mathcal{K}$ the subset of $\mathrm{GL}(2; \mathbb{Z})$ consisting of diagonal matrices. For each $s \in \{1, \dots, \ell\}$ there is $K_s \in \mathcal{K}$ such that $K_s(\widetilde{\boldsymbol{d}}^s) = \boldsymbol{d}^s$. Set $K_0 = \mathtt{id}$. Fix $s \in \{1, \dots, \ell\}$. Denote $E_s' = K_s E_s K_{s-1}$. Since $K_{s-1}^{-1} = K_{s-1}$, we have $\boldsymbol{d}^s = E_s'(\boldsymbol{d}^{s-1})$. Since $\mathcal{E}$ is invariant under multiplication from the left and the right by elements of $\mathcal{K}$, we have $E_s' \in \mathcal{E}$. Because $\boldsymbol{d}^{s-1}, \boldsymbol{d}^s \in \mathbb{R}_2^+$, each

transformation $E \in \mathcal{E}$ that maps $\boldsymbol{d}^{s-1}$ to $\boldsymbol{d}^s$ coincides (possibly after precomposing or/and postcomposing it with the involution $(x,y) \xmapsto{I} (y,x)$) with one of following three:

$$(b,c) \longmapsto (b+c,c), \quad (b,c) \longmapsto (b-c,c), \quad (b,c) \longmapsto (c-b,c).$$

In particular, this is true for $E = E'_s$. The path from $\boldsymbol{d}^{s-1}$ to $\boldsymbol{d}^s$ can therefore be chosen to be one of the following admissible paths (with a possible addition of the involution I at the beginning or/and at the end):

$$\begin{aligned} &(b,c) \xmapsto{P_{12}} (b+c,c), \\ &(b,c) \xmapsto{M_{12}} (b-c,c), \\ &(b,c) \xmapsto{M_{21}} (b,c-b) \xmapsto{I} (c-b,b) \xmapsto{P_{21}} (c-b,c). \end{aligned}$$

This proves Lemma B.5. □

We shall prove now that there exists a low admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$. We call an admissible path $\boldsymbol{d} = \boldsymbol{d}^0, \ldots, \boldsymbol{d}^\ell = \boldsymbol{e}$ special if each of the moves from $\boldsymbol{d}^{s-1}$ to $\boldsymbol{d}^s$ is by one of the following three operators: $P = P_{12}, M = M_{12}, I = I_{12}$. Since $P_{21} = IP_{12}I$ and $M_{21} = IM_{12}I$, every admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$ can be transformed into a special one. In particular, special admissible paths exist. Let $\mathfrak{p}$ be a special admissible path of minimal length. We claim that $\mathfrak{p}$ is low.

Assume first that some P move precedes some M move in the path $\mathfrak{p}$. Then the path must contain either the sequences of moves P, M or the sequences of moves P, I, M (note that adjacent I moves cannot occur due to the minimality of $\mathfrak{p}$). In the first case, the sequence

$$(c,d) \xmapsto{P} (c+d,d) \xmapsto{M} (c,d)$$

can be removed from the path, in contradiction with the minimality of $\mathfrak{p}$. The second case,

$$(c,d) \xmapsto{P} (c+d,d) \xmapsto{I} (d,c+d) \xmapsto{M} (-c,d),$$

is impossible because $-c$ is negative. It remains to consider the case where for some s each of the first s moves is either M or I, and each of the last $\ell - s$ moves is either P or I. Then we have

$$\boldsymbol{d} \geq \boldsymbol{d}^1 \geq \cdots \geq \boldsymbol{d}^s \leq \cdots \leq \boldsymbol{d}^{\ell-1} \leq \boldsymbol{e},$$

and the path $\mathfrak{p}$ is low. □

Lemma B.6. *Let $\boldsymbol{d} = (d_1, \ldots, d_k)$, $\boldsymbol{e} = (e_1, \ldots, e_k)$ be vectors in $\mathbb{R}^k_+$ satisfying $\langle \boldsymbol{d} \rangle = \langle \boldsymbol{e} \rangle$. Assume that there is $i \in \{1, \ldots, k\}$ such that $d_i = e_i$, $d_i \leq d_j$ for all j, and d_i is primitive (indivisible) in $\langle \boldsymbol{d} \rangle$. Then there is a low admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$.*

Proof. Assume for notational convenience that $i = k$. By repeatedly subtracting the number $d_k = e_k$ from the components of $\boldsymbol{d}$ that exceed d_k, we construct an admissible path $\boldsymbol{d}, \boldsymbol{d}^1, \ldots, \boldsymbol{d}^\ell$, where $\boldsymbol{d} \geq \boldsymbol{d}^1 \geq \cdots \geq \boldsymbol{d}^\ell$ and $\boldsymbol{d}^\ell$ is such that $d^\ell_k \geq d^\ell_j$ for all j. Using the

same procedure, we obtain an admissible path $\boldsymbol{e}, \boldsymbol{e}^1, \ldots, \boldsymbol{e}^m$, where $\boldsymbol{e} \geq \boldsymbol{e}^1 \geq \cdots \geq \boldsymbol{e}^m$ and $\boldsymbol{e}^m$ is such that $e^m_k \geq e^m_j$ for all j.

Denote by π the projection $\mathbb{R}^k \to \mathbb{R}^{k-1}$ that forgets the k-th component, and define an equivalence relation on $\mathbb{R}^k$ as follows. Let $\boldsymbol{u} = (u_1, \ldots, u_k)$, $\boldsymbol{v} = (v_1, \ldots, v_k)$. Then $\boldsymbol{u} \sim \boldsymbol{v}$ if

- $\diamond$ $\langle \boldsymbol{u} \rangle = \langle \boldsymbol{v} \rangle$,
- $\diamond$ $u_k = v_k$,
- $\diamond$ there is an operator $A \in \mathrm{GL}(k-1, \mathbb{Z})$ such that all components of the vector $A(\pi(\boldsymbol{u})) - \pi(\boldsymbol{v})$ are integer multiples of u_k.

We claim that $\boldsymbol{d} \sim \boldsymbol{e}$. Indeed, since d_k is indivisible in $\langle \boldsymbol{d} \rangle$, there is a subgroup $\Lambda \subset \langle \boldsymbol{d} \rangle$ such that $\langle \boldsymbol{d} \rangle = \langle d_k \rangle \oplus \Lambda$. For each $j \in \{1, \ldots, k-1\}$ there exists $d'_j \in \Lambda$ (resp. $e'_j \in \Lambda$) that differs from d_j (resp. e_j) by an integer multiple of d_k. Consider the vectors $\boldsymbol{d}' = (d'_1, \ldots, d'_{k-1})$, $\boldsymbol{e}' = (e'_1, \ldots, e'_{k-1})$ in $\mathbb{R}^{k-1}$. The groups $\langle \boldsymbol{d}' \rangle$ and $\langle \boldsymbol{e}' \rangle$ are subgroups of Λ. Since $\langle d_k \rangle \oplus \Lambda = \langle d_k \rangle + \langle \boldsymbol{d}' \rangle = \langle d_k \rangle + \langle \boldsymbol{e}' \rangle$, we have $\langle \boldsymbol{d}' \rangle = \langle \boldsymbol{e}' \rangle = \Lambda$. By Lemma B.2, there is $A \in \mathrm{GL}(k-1, \mathbb{Z})$ that takes $\boldsymbol{d}'$ to $\boldsymbol{e}'$. This A has the required property.

By construction, we have $\boldsymbol{d} \sim \boldsymbol{d}^\ell$ and $\boldsymbol{e} \sim \boldsymbol{e}^m$. Thus $\boldsymbol{d}^\ell \sim \boldsymbol{e}^m$. For a positive number C, we say that an admissible path has height $\leq C$ if the components of every vector involved in this path do not exceed C. We shall construct an admissible path of height $\leq d_k$ from $\boldsymbol{d}^\ell$ to $\boldsymbol{e}^m$. By concatenating the path $\boldsymbol{d}, \ldots, \boldsymbol{d}^\ell$, the path from $\boldsymbol{d}^\ell$ to $\boldsymbol{e}^m$, and the path $\boldsymbol{e}^m, \ldots, \boldsymbol{e}$, we then obtain a low admissible path from $\boldsymbol{d}$ to $\boldsymbol{e}$.

Let $A \in \mathrm{GL}(k-1, \mathbb{Z})$ be such that the components of the vector $A(\pi(\boldsymbol{d}^\ell)) - \pi(\boldsymbol{e}^m)$ are integer multiples of d_k. Denote by Q_j, $j \in \{1, \ldots, k-1\}$, the operator in $\mathrm{GL}(k-1, \mathbb{Z})$ that changes the sign of the j-th component and keeps all other components intact. Each operator in $\mathrm{GL}(k-1, \mathbb{Z})$ can be written as a product of operators Q_j, I_{ij}, and P_{ij}, see e.g. [5, p. 43]. In particular, we have $A = A_r \cdots A_2 A_1$, where either $A_s = Q_{j_s}$, or $A_s = I_{i_s j_s}$, or $A_s = P_{i_s j_s}$ for each $s \in \{1, \ldots, r\}$. Denote $\boldsymbol{v}^0 = \pi(\boldsymbol{d}^\ell)$, $\boldsymbol{v}^s = A_s \cdots A_1(\boldsymbol{v}^0)$. Consider the map $\psi\colon \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}_+$ that takes a vector $(x_1, \ldots, x_{k-1})$ to the vector $(y_1, \ldots, y_{k-1})$, where $y_j \in \,]0, d_k]$ and $x_j - y_j$ is an integer multiple of d_k for each j. Denote $\boldsymbol{u}^s = \psi(\boldsymbol{v}^s)$. Then $\boldsymbol{u}^0 = \pi(\boldsymbol{d}^\ell)$, $\boldsymbol{u}^r = \pi(\boldsymbol{e}^m)$, and, since $\psi \circ A_s \circ \psi = \psi \circ A_s$, we have $\boldsymbol{u}^s = \psi(A_s \boldsymbol{u}^{s-1})$ for each $s \in \{1, \ldots, r\}$.

Denote by $\boldsymbol{w}^s = (w^s_1, \ldots, w^s_{k-1}, d_k)$ the vector in $\mathbb{R}^k_+$ obtained by complementing the $k-1$ components of the vector $\boldsymbol{u}^s$ with the k-th component equal to d_k. We have $\boldsymbol{w}^0 = \boldsymbol{d}^\ell$, $\boldsymbol{w}^r = \boldsymbol{e}^m$, and $\boldsymbol{w}^s \leq d_k$ for each s. It suffices to prove that there is an admissible path of height $\leq d_k$ from $\boldsymbol{w}^{s-1}$ to $\boldsymbol{w}^s$ for each s.

If $A_s = I_{ij}$, then $\boldsymbol{w}^s = I_{ij}(\boldsymbol{w}^{s-1})$ and such a path consists of the single step I_{ij}. Let $A_s = Q_j$. For notational convenience, assume that $k = 2$, $j = 1$. Let $\boldsymbol{w}^{s-1} = (c, d)$. If $c = d$, then $\boldsymbol{w}^s = \boldsymbol{w}^{s-1}$, and there is nothing to prove. If $c < d$, then $\boldsymbol{w}^s = (d-c, d)$, and the path

$$(c, d) \xmapsto{M_{21}} (c, d-c) \xmapsto{I_{12}} (d-c, c) \xmapsto{P_{21}} (d-c, d) \tag{B1}$$

is of height $\leq d$.

Let $A_s = P_{ij}$. If $w_i^s + w_j^s \leq d_k$, then $\boldsymbol{w}^s = P_{ij}(\boldsymbol{w}^{s-1})$ and the required path consists of the step P_{ij}. Let $w_i^s + w_j^s > d_k$. For notational convenience, assume that $k = 3$, $i = 1$, $j = 2$. We have $\boldsymbol{w}^{s-1} = (b, c, d)$, where $b + c > d$. If $c = d$, then $\boldsymbol{w}^s = \boldsymbol{w}^{s-1}$, and there is nothing to prove. If $c < d$, then $\boldsymbol{w}^s = (b+c-d, c, d)$. By (B1), there is an admissible path of height $\leq d$ from $\boldsymbol{w}^{s-1}$ to $\boldsymbol{w}' = (b, d-c, d)$, and hence to $\boldsymbol{w}'' = M_{12}(\boldsymbol{w}) = (b+c-d, d-c, d)$. By (B1) again, there is an admissible path of height $\leq d$ from $\boldsymbol{w}''$ to $\boldsymbol{w}^s$. Lemma B.6 is proved. □

Proof of Theorem B.1. The proof is by induction on k. Lemma B.3 and Lemma B.4 prove the statement for $k = 1$ and $k = 2$. We shall prove the statement for $k \geq 3$ assuming that it holds for $k - 1$. In view of Lemma B.3, we can assume that $\operatorname{rk}\langle \boldsymbol{d} \rangle \geq 2$.

Lemma B.7. *Let $\boldsymbol{u} = (u_1, \ldots, u_k) \in \mathbb{R}_+^k$. There is a low admissible path from $\boldsymbol{u}$ to a vector $\boldsymbol{u}^+ = (u_1^+, \ldots, u_k^+) \in \mathbb{R}_+^k$ such that $\boldsymbol{u}^+ \leq \boldsymbol{u}$, $u_k^+ \leq u_j^+$ for all j, and u_k^+ is indivisible in $\langle \boldsymbol{u} \rangle$.*

Proof. By repeatedly subtracting from the last component of the vector $\boldsymbol{u}$ its other components, we construct a low admissible path from $\boldsymbol{u}$ to a vector $\boldsymbol{u}' = (u_1, \ldots, u_{k-1}, u_k') \in \mathbb{R}_+^k$ such that $\boldsymbol{u}' \leq \boldsymbol{u}$ and $u_k' \leq u_j$ for all j. If u_k' is indivisible in $\langle \boldsymbol{u} \rangle$, then $\boldsymbol{u}^+ = \boldsymbol{u}'$, and the lemma is proved. Denote $\boldsymbol{u}^- = (u_1, \ldots, u_{k-1})$. If u_k' is not primitive, then the rank m of the free Abelian group $\langle \boldsymbol{u}^- \rangle$ is at least 2. We claim that there is a basis $x_1, \ldots, x_m$ of $\langle \boldsymbol{u}^- \rangle$ formed by positive numbers smaller than u_k'. Indeed, since $\operatorname{rk}\langle \boldsymbol{u}^- \rangle = m \geq 2$, there is a primitive element $x_1 \in \langle \boldsymbol{u}^- \rangle$ satisfying $0 < x_1 < u_k'$. Extend x_1 to a basis $x_1, \ldots, x_m$ of $\langle \boldsymbol{u}^- \rangle$, and add to each x_j, $2 \leq j \leq m$, a suitable integer multiple of x_1 to ensure that $0 < x_j < x_1$. After reordering the elements of the basis, we can assume that x_m is the smallest of them. Consider the vector $\boldsymbol{v}^- = (x_1, \ldots, x_1, x_2, \ldots, x_m)$ in $\mathbb{R}_+^{k-1}$, and the vectors $\boldsymbol{v} = (x_1, \ldots, x_1, x_2, \ldots, x_m, u_k')$, $\boldsymbol{u}^+ = (x_1, \ldots, x_1, x_2, \ldots, x_{m-1}, u_k', x_m)$ in $\mathbb{R}_+^k$. By the induction hypothesis, there is a low admissible path from $\boldsymbol{u}^-$ to $\boldsymbol{v}^-$. Attaching to the vectors of this path u_k' as the k-th component, we construct a low admissible path from $\boldsymbol{u}'$ to $\boldsymbol{v}$. We have $\boldsymbol{v} \leq \boldsymbol{u}' \leq \boldsymbol{u}$ and $\boldsymbol{u}^+ \leq \boldsymbol{u}$. Therefore, concatenating the path from $\boldsymbol{u}$ to $\boldsymbol{u}'$, the path from $\boldsymbol{u}'$ to $\boldsymbol{v}$, and the one-step transposition path from $\boldsymbol{v}$ to $\boldsymbol{u}^+$, we obtain the required low admissible path. □

By Lemma B.7, we can assume that d_k and e_k are indivisible in $\langle \boldsymbol{d} \rangle$, $d_k \leq d_j$ and $e_k \leq e_j$ for all j. If $d_k = e_k$, then Lemma B.6 (with $i = k$) proves the statement.

Otherwise, $\langle d_k, e_k \rangle$ is a free Abelian group of rank 2. Consider the elements $x \in \langle \boldsymbol{d} \rangle$ such that $n_x x \in \langle d_k, e_k \rangle$ for some non-zero integer n_x. They form a rank 2 free Abelian subgroup Δ of $\langle \boldsymbol{d} \rangle$. Then $\langle \boldsymbol{d} \rangle = \Delta \oplus \Lambda$ for some free Abelian subgroup $\Lambda \in \langle \boldsymbol{d} \rangle$. Since d_k and e_k are indivisible in Δ, there are $d_{k-1}', e_{k-1}' \in \Delta$ such that $\Delta = \langle d_k, d_{k-1}' \rangle = \langle e_k, e_{k-1}' \rangle$. After adding to d_{k-1}' (resp. e_{k-1}') an integer multiple of d_k (resp. e_k), we can assume that $0 < d_{k-1}' < d_k$ and $0 < e_{k-1}' < e_k$. Pick a basis $y_1, \ldots, y_m$ of the free Abelian group Λ. Consider the vectors $\boldsymbol{d}' = (y_1, \ldots, y_1, y_2, \ldots, y_m, d_{k-1}', d_k)$, $\boldsymbol{e}' = (y_1, \ldots, y_1, y_2, \ldots, y_m, e_{k-1}', e_k)$ in $\mathbb{R}^k$. We have $\langle \boldsymbol{d} \rangle = \langle \boldsymbol{d}' \rangle = \langle \boldsymbol{e}' \rangle$. After adding to the numbers y_j suitable integer multiples of

$\min(d'_{k-1}, e'_{k-1})$ and reordering them, we can achieve that $0 < y_1 < \cdots < y_m < d'_{k-1} < d_k$ and $0 < y_1 < \cdots < y_m < e'_{k-1} < e_k$, still preserving the condition $\langle \boldsymbol{d} \rangle = \langle \boldsymbol{d}' \rangle = \langle \boldsymbol{e}' \rangle$.

Applying Lemma B.6 to the pairs $\boldsymbol{d}, \boldsymbol{d}'$ and $\boldsymbol{e}, \boldsymbol{e}'$, with $i = k$, we find a low admissible path $\mathfrak{p}_0$ connecting $\boldsymbol{d}$ to $\boldsymbol{d}'$ and a low admissible path $\mathfrak{p}_1$ connecting $\boldsymbol{e}'$ to $\boldsymbol{e}$. Since y_1 is indivisible in $\langle \boldsymbol{d} \rangle$, we can apply Lemma B.6 to the pair $\boldsymbol{d}', \boldsymbol{e}'$, with $i = 1$, and find a low admissible path $\mathfrak{p}$ that connects $\boldsymbol{d}'$ to $\boldsymbol{e}'$. Since $\boldsymbol{d}' \leq \boldsymbol{d}$ and $\boldsymbol{e}' \leq \boldsymbol{e}$, the concatenation of $\mathfrak{p}_0$, $\mathfrak{p}$, and $\mathfrak{p}_1$ is a low admissible path. This completes the proof of Theorem B.1. □

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